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Bounds on the Capacity of Weakly Constrained Two-Dimensional Codes

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Abstract — Upper and lower bounds are presented for the capacity of weakly constrained two-dimensional codes. The maximum entropy is calculated for two simple models of 2-D codes constraining the probability of neighboring 1s as an example. For given models of the coded data, upper and lower bounds on the capacity for 2-D channel models based on occurrences of neighboring 1s are considered.

I. INTRODUCTION

Weakly constrained codes in 1-D [1] and constrained codes in 2-D [2] have been considered. We define weakly constrained codes in 2-D, by constraining the values of the probability of subsets of N by M configurations to lie in a given interval.

Example 1. A max. probability, \( p_{\text{max}} \), is imposed for the occurrence of two neighboring 1’s on a 2-D set of binary values.

II. BOUNDS ON CAPACITIES AND ENTROPIES

Let \( X \) and \( Y \) denote the stochastic variables describing the coded data written or sent and the data received, respectively. The achievable rate of the code \( X \) over the 2-D channel is given by the mutual entropy

\[
I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)
\]

where \( H() \) and \( H(|) \) denotes the entropy and conditional entropy, respectively. Maximizing \( I(X;Y) \) over the code \( X \), given the channel statistics, defines the 2-D channel capacity.

In 1-D, \( Y \) is a function of a Markov process for which the entropy \( H(Y) \) may be bounded [3]. This approach also yields bounds on \( H(X|Y) \). In 2-D we consider the class of fields \( X \) for which \( k \) consecutive rows may be described by a vector Markov process, e.g. Pickard Random Fields (PRF) [4], or a \( k \)-dimensional vector function of a \( k' \)-dimensional vector Markov process, e.g. as in [2].

Given a vector Markov process \( X \), \( H(X) \) and \( H(Y|X) \) may be calculated. The other terms of (1) may be bounded based on applying the 1-D bound [3] to the vector processes. This involves the difference of the entropies on \( N \) and \( N-1 \) rows and generalization of the bound to 2-D. Two examples of bounds are given below based on vector Markov processes. A lower bound based on (1) and bounding \( H(X|Y) \) using an \( m + l + 1 \) segment \( (Y_{t-l}^{t+l}) \) of \( k \)-row vectors, \( Y_t \), of \( Y \) is given by

\[
I(X,Y) \geq H(X) - H(X|Y) = I(X,Y) \leq H(Y(F(Y^{t+1})) - H(Y|X)
\]

where \( F(Y^{t+1}) \) is the mapping of the causal part with respect to element \( Y_t \), e.g. defined by a subset. \( H(Y|F(Y^{t+1})) \) may be bounded using the 1-D result [3]. For a given weak constraint and model for \( X \), the bounds (2-3) may be optimized over the free parameters of \( X \).

For a given constraint defined on the probability of occurrence of configurations the capacity of the code, \( \max H(X) \) may be bounded by letting a Lagrangian control the probability of a constrained configuration. As for 2-D hard constraints, a band source of width \( m \) and extending vertically is introduced defining states having \( N - 1 \) by \( m \) elements. Each transition specifies an \( N \) by \( m \) rectangle by combining the starting and ending states. Let \( H(m) \) denote the entropy given by generating \( m \) new elements with each transition. An upper bound (on \( \max H(X) \)) is given by \( H(m)/m \) optimized under the given constraint. A lower bound is obtained by concatenating bands of width \( m \) now with the additional constraint that the weak constraint is still satisfied after the concatenation of independent bands.

III. EXPERIMENTS

In [5], a variation of the PRF was introduced. The probabilities are derived from a 1-D binary Markov chain. For the weak constraint of Ex. 1 and given values of \( p_{\text{max}} \), we present the values of entropy \( H(X) \) optimized over the parameters of these two models. The MC-based model yielded a slightly higher entropy. Compared with upper and lower bounds calculated using the Lagrangian techniques all the values were close for the same value of \( p_{\text{max}} \).

Let the input and output values be binary. As a simple model of the 2-D channel, related to the weak constraint of Ex. 1, we define the error probability to be a function of the neighboring 1’s in the input. We present the bounds (2-3) obtained for different functions and different levels of errors, modeling \( X \) by the MC-based model [5], as iid binary data and imposing a hard constraint \( (p_{\text{max}} = 0) \). The optimized MC-based codes of course yields the highest capacity. The noise level has to be quite significant before the hard constraint code outperforms the iid code.

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