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Optimal random perturbations for stochastic approximation using a simultaneous perturbation gradient approximation

Sadegh, Payman; Spall, J. C.

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Using Lemma 3 and a similar argument we have that

$$\|S_{11} - S_{12}Q_{1/\delta}S_{21}\|_2 \leq \mu + \epsilon/2 + [\bar{\kappa}_{11} + (\bar{\kappa}_{21}\underline{\kappa}\|S_{12}\|_\infty$$
$$+ \bar{\kappa}_{12}\|S_{21}\|_\infty)(1 + \|T_{11}\|_\infty)](1 - \delta).$$

Selecting δ such that

$$[\kappa_{11} + (\kappa_{21} + \kappa_{12})(1 + \|T_{11}\|_\infty)](1 - \delta) < 1 - \gamma$$

and

$$[\bar{\kappa}_{11} + (\bar{\kappa}_{21}\underline{\kappa}\|S_{12}\|_\infty + \bar{\kappa}_{12}\|S_{21}\|_\infty)(1 + \|T_{11}\|_\infty)](1 - \delta) \leq \epsilon/2$$

we have that $\|T_{11,\delta} - T_{12,\delta}QT_{21,\delta}\|_\infty < 1$ and $\|S_{11} - S_{12}Q_{1/\delta}S_{21}\|_2 \leq \mu$. Thus $Q_{1/\delta}$ is feasible for (B2) and achieves a performance of at most $\mu + \epsilon$. This establishes the claim and the theorem, since necessarily $\mu_\delta \leq \mu + \epsilon$.

ACKNOWLEDGMENT

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Optimal Random Perturbations for Stochastic Approximation Using a Simultaneous Perturbation Gradient Approximation

Payman Sadegh and James C. Spall

Abstract— The simultaneous perturbation stochastic approximation (SPSA) algorithm has recently attracted considerable attention for challenging optimization problems where it is difficult or impossible to obtain a direct gradient of the objective (say, loss) function. The approach is based on a highly efficient simultaneous perturbation approximation to the gradient based on loss function measurements. SPSA is based on picking a simultaneous perturbation (random) vector in a Monte Carlo fashion as part of generating the approximation to the gradient. This paper derives the optimal distribution for the Monte Carlo process. The objective is to minimize the mean square error of the estimate. The authors also consider maximization of the likelihood that the estimate be confined within a bounded symmetric region of the true parameter. The optimal distribution for the components of the simultaneous perturbation vector is found to be a symmetric Bernoulli in both cases. The authors end the paper with a numerical study related to the area of experiment design.

Index Terms— Experiment design, optimal probability distribution, optimization, SPSA, stochastic approximation.

I. INTRODUCTION

Consider the problem of determining the value of a p-dimensional parameter vector to minimize a loss function $L(\theta)$, where only measurements of the loss function are available (i.e., no gradient information is directly available). The simultaneous perturbation stochastic approximation (SPSA) algorithm has recently attracted considerable attention for challenging optimization problems of this type in application areas such as adaptive control, pattern recognition, discrete-event systems, neural network training, and model parameter estimation; see, e.g., [1]–[6].

SPSA was introduced in [7] and more thoroughly analyzed in [8]. The essential feature of SPSA—which accounts for its power and relative ease of use in challenging multivariate optimization problems—is the underlying gradient approximation that requires only two loss function measurements, regardless of the number of parameters being optimized. Note the contrast of two function measurements with the $2p$ measurements required in classical finite difference-based approaches (i.e., the Kiefer–Wolfowitz SA algorithm). Under reasonably general conditions, it was shown in [8] that the p-

fold savings in function measurements per gradient approximation can translate directly into a p-fold savings in a total number of measurements needed to achieve a given level of accuracy in the optimization process. This means that the SPSA approach uses the same number of iterations as the finite difference approach to achieve a given level of mean square error (MSE) in the optimization process, but each iteration of SPSA uses only $1/p$ the number of function measurements.

An essential part of the gradient approximation is a simultaneous (random) perturbation relative to the current estimate of θ. This perturbation is generated in a Monte Carlo fashion as part of the optimization process. Since the user has complete control over the perturbation distribution, there is strong reason to choose a distribution as a means of minimizing the number of (potentially costly) function measurements needed in the optimization process. These function measurements may involve physical experiments involving labor or material costs as well as computer-related costs associated with simulations or data processing.

The aim of this paper is to determine the form of the optimal distribution for the simultaneous perturbations. This will involve both analytical analysis based on the asymptotic properties of the parameter iterate and numerical finite sample experimentation. The related objectives considered here are to minimize the MSE of the estimate and to maximize the likelihood that the parameter iterate is restricted to a symmetric bounded region around the true parameter.

The rest of the paper is organized as follows. In Section II, we briefly review the SPSA algorithm and present the problem formulation. Section III considers the choice of random perturbations. In Section IV, we study an optimization problem from the area of statistical experiment design for dynamic system identification. Section V offers concluding remarks.

II. PROBLEM FORMULATION

Consider the problem of finding a root θ^* of $g(\theta) \equiv \partial L(\theta)/\partial \theta = 0$ for some differentiable loss function $L : \mathbb{R}^p \to \mathbb{R}$. In the case where the dependence of the loss function upon θ is unknown, but the loss function is observed in the presence of noise, an SA algorithm of the generic Kiefer–Wolfowitz type (see, e.g., [9]) is appropriate.

Let us now briefly review the SPSA algorithm (see [8]) for the problem posed above. Let $\hat{\theta}_k$ denote the estimate for θ at the kth iteration. The SPSA algorithm has the form

$$\hat{\theta}_{k+1} = \hat{\theta}_k - a_k \hat{g}_k(\hat{\theta}_k)$$

where $\{a_k\}$ is a gain sequence and $\hat{g}_k(\hat{\theta}_k)$ is a simultaneous perturbation approximation to $g(\hat{\theta}_k)$ at iteration k. The simultaneous perturbation approximation is defined as follows. Let $\Delta_k \in \mathbb{R}^p$ be a vector of p mutually independent mean zero random variables $\{\Delta_{k1}, \Delta_{k2}, \cdots, \Delta_{kp}\}$. Consistent with the usual framework of stochastic approximations, we have noisy measurements of the loss function at specified "design levels." In particular, at the kth iteration

$$y_k^{(+)} = L(\hat{\theta}_k + c_k \Delta_k) + \epsilon_k^{(+)}$$
$$y_k^{(-)} = L(\hat{\theta}_k - c_k \Delta_k) + \epsilon_k^{(-)}$$

where $\{c_k\}$ is a gain sequence and $\epsilon_k^{(+)}$ and $\epsilon_k^{(-)}$ represent measurement noise terms. The basic simultaneous perturbation form for the estimate of $g(.)$ at the kth iteration is then

$$\hat{g}_k(\hat{\theta}_k) = \begin{bmatrix} \dfrac{y_k^{(+)} - y_k^{(-)}}{2c_k \Delta_{k1}} \\ \vdots \\ \dfrac{y_k^{(+)} - y_k^{(-)}}{2c_k \Delta_{kp}} \end{bmatrix}. \tag{1}$$

Note that at each iteration, only *two* measurements are needed to form the estimate. To help mitigate noise effects in high noise environments, it is sometimes useful to consider averaging among gradient approximations, each generated as in (1) based on a new pair of measurements that are conditionally (on $\hat{\theta}_k$), independent of the other measurement pairs; this is examined in [8] but will not be examined further here. In addition to the usual Kiefer–Wolfowitz SA assumptions (see [10]), we assume throughout the paper that:

A1) $a_k = a/k^\alpha$, and $c_k = 1/k^\gamma$ where $a > 0, 0 < \alpha \leq 1, \gamma > 0$, $\alpha - \gamma > 0.5$, $\alpha - 2\gamma > 0$, and $3\gamma - \alpha/2 \geq 0$ (since c_k and Δ_k always appear together as $c_k \Delta_k$, we fix the numerator in c_k to unity and let Δ_k vary freely).

A2) $E\{\epsilon_k^{(+)} - \epsilon_k^{(-)} | \hat{\theta}_k, \Delta_k\} = 0$, and for some $\alpha_0, \delta > 0$ and $\forall k$, $E\{\epsilon_k^{\pm (2+\delta)}\} < \alpha_0$. Moreover, there is a σ^2 such that $E\{(\epsilon_k^{(+)} - \epsilon_k^{(-)})^2 | \hat{\theta}_k, \Delta_k\} \to \sigma^2$ as $k \to \infty$.

A3) For all $k < \infty$, $\{\Delta_{ki}\}$ $(i = 1, \cdots, p)$ are i.i.d. and symmetrically distributed about zero with $|\Delta_{ki}| \leq \alpha_0$ a.s. and $E|\Delta_{ki}^{-1}| \leq \alpha_1$ a.s. for some $\alpha_0, \alpha_1 > 0$. For some $\alpha_2, \alpha_3, \delta > 0$, it holds that $E\{|L(\hat{\theta}_k \pm c_k \Delta_k)|^{2+\delta}\} \leq \alpha_2$ and $E(\Delta_{ki}^{-2-\delta}) \leq \alpha_3, i = 1, \cdots, p$. Moreover, there are ρ^2, ξ^2 such that as $k \to \infty$, $E(\Delta_{ki}^{-2}) \to \rho^2$ and $E(\Delta_{ki}^2) \to \xi^2$ for all $i = 1, \cdots, p$.

A4) For almost all $\hat{\theta}_k$, there is an open ball about $\hat{\theta}_k$ whose radius is independent of k or $\hat{\theta}_k$, where the third derivative of the loss function exists continuously and is uniformly bounded.

The reader is referred to [8] for more details and remarks on the assumptions.

The problem of selecting random perturbations is formulated as selecting a sequence of probability distributions for Δ_{ki}, $k = 1, 2, \cdots$, each from the set of allowable probability distributions for the random perturbations [see A3)]. The objective is to optimize a suitable criterion related to the parameter estimate.

For small k, the exact distribution of $\hat{\theta}_k$ is dependent upon the unknown joint probability distribution of the noise sequence. Therefore, we solve the optimal random perturbation problem using the asymptotic distribution of the estimate. It follows from [8, Proposition 2] that as $k \to \infty$

$$k^{\beta/2}(\hat{\theta}_k - \theta^*) \overset{\text{dist}}{\to} Z \sim N(\xi^2 d, \rho^2 D) \tag{2}$$

where β is a positive constant, and d and D are quantities *not* dependent upon the random perturbations. The matrix D depends on the Hessian of $L(\theta)$ at θ^* and σ^2, and d depends on the third-order derivative of $L(\theta)$ at θ^*. Both d and D are dependent upon a, α, and γ. The reader is referred to [8] for the detailed forms of d and D.

From (2), it is evident that the distribution of Z is affected by the random perturbations only through ρ^2 and ξ^2 [see A3)]. Hence, using the asymptotic result for sufficiently large number of iterations, the problem simplifies to selection of a *single* probability distribution for Δ_{ki}, for *all* $k = 1, 2, \cdots$, optimizing some criterion related to Z.

III. OPTIMAL CHOICE OF RANDOM PERTURBATIONS

In this section, we study the selection of random perturbations with the aim of minimizing (the trace of) the MSE of the estimate (Section III-A), and maximizing the probability of restricting the estimation error within some bounded symmetric about zero region (Section III-B), respectively. The analysis here is based on the asymptotic distribution of the parameter iterate; the authors are unaware of any corresponding finite sample result that would be useful in such calculations.

Throughout Sections III-A and III-B, we assume that $d \neq 0$. The degenerate case $d = 0$ will be discussed in Remark 2.

A. MSE

We refer to MSE $\triangleq E\{\mathrm{trace}[ZZ^\top]\}$ as the mean square error criterion. Using (2)

$$\mathrm{MSE} = \rho^2 \,\mathrm{trace}\{D\} + \xi^4\, d^\top d. \qquad (3)$$

In the following, we denote $K_1 = \mathrm{trace}\{D\}$ and $K_2 = d^\top d$ (note that K_1 and K_2 do not depend upon the random perturbations). Obviously, the distributions of interest contain at least two support points and are symmetric [see A3)]. Proposition 1 below shows that regardless of the values of K_1 and K_2, for any allowable perturbation distribution containing more than two support points, a lower value of MSE is provided by a distribution that contains only two support points.

Proposition 1: The allowable perturbation distribution for Δ_{ki}, $k = 1, 2, \cdots$ and $i = 1, 2, \cdots, p$, minimizing the mean square error criterion is necessarily a symmetric Bernoulli distribution.

Proof: Let ξ^2 and ρ^2 [as defined in A3)] correspond to a symmetric distribution P for Δ_{ki} where P contains more than two support points. According to the Schwarz inequality $\rho^2 \geq 1/\xi^2$. Now, $E(\Delta_{ki}^2)E(1/\Delta_{ki}^2) = 1$ iff Δ_{ki} and $1/\Delta_{ki}$ are constant multiples of one another in each realization, and for symmetrically distributed variables that holds iff Δ_{ki} is symmetric Bernoulli distributed. Hence $\rho^2 > 1/\xi^2$ and $K_1\rho^2 + K_2\xi^4 > K_1\rho^2 + K_2(1/\rho^2)^2$, and the necessity follows since the right-hand side of the inequality is equal to the MSE for the Bernoulli distribution $\Delta_{ki} = \pm\rho^{-1}$. Notice that the result does not depend on K_1 and K_2. $\square$

Remark 1: For known K_1 and K_2, Proposition 1 can be used to derive an analytic expression for the optimal perturbation distribution. Proposition 1 implies that $\rho\xi = 1$ for the optimal distribution. Inserting $\rho = 1/\xi$ in (3), it follows immediately that the minimizing argument of MSE with respect to ξ is equal to $(K_1/2K_2)^{1/6}$, i.e., the Bernoulli distribution

$$\Delta_{ki} = \pm\left(\frac{K_1}{2K_2}\right)^{1/6} \qquad (4)$$

is the unique optimizer of the mean square error criterion. This is analogous to the calculations for the optimal gain sequences of stochastic algorithms; see, e.g., [11] and [12]. To invoke the optimality result given by (4), it is obviously required to compute K_1 and K_2 using an *a priori* model for $L(\theta)$, and in most practical cases, it is difficult to specify such *a priori* models. The situation can be partially mitigated by assuming an *a priori implicit* model for $L(\theta)$ (i.e., a model that allows computation of $L(\theta)$ for each θ). In such cases, it is often difficult to accurately evaluate the second- and third-order derivatives or the noise variance σ^2, which are required for the calculation of K_1 and K_2. The following procedure may be useful in such situations. By applying SPSA to the available implicit model using very large number of iterations K, we obtain the estimate $\hat{\theta}_K$ which we use as the true optimum in our calculations. We then approximate K_1 and K_2, using the given model and $\hat{\theta}_K$, and use (4) to find an approximation to the optimal perturbation magnitude which will be used as the initial guess for a numerical search. Proposition 1 implies that the optimal perturbation distribution should be sought among symmetric Bernoulli distributions. We sample the Δ_{ki} from Bernoulli distributions with varying magnitudes around the initial guess. For each magnitude, we apply SPSA a number of times (cross sections), obtain $\hat{\theta}_k$ for each cross section to find $\|\hat{\theta}_k - \hat{\theta}_K\|^2$ where $k \ll K$ is some large iteration number of interest, and average over the computed values of $\|\hat{\theta}_k - \hat{\theta}_K\|^2$ to numerically evaluate the mean square error for each one of the Bernoulli distributions, respectively. The numerical study of the paper illustrates such a procedure. It is important to note that SPSA only requires noisy evaluations of the loss function (typically obtained by real experimentations). The given *a priori* models are then only used for the optimal design of random perturbations.

B. Probability Criterion

Our objective in this subsection is to maximize the probability of restricting the error Z within some bounded symmetric (about zero) region denoted by V_θ. A similar approach is pursued in [13] to determine the constants of a Robbins–Monro-type stochastic approximation algorithm. Denoting probability by $\mathrm{Pr}(\cdot)$, the optimality criterion is written as

$$J = \mathrm{Pr}(Z \in V_\theta). \qquad (5)$$

An important special case for J is where V_θ is the closed unit ball. Then the criterion to be maximized is $\mathrm{Pr}(\|Z\| \leq A)$, where as usual, $\|\cdot\|$ denotes Euclidean norm and A is a positive number chosen by the user. It reflects the user's tolerable amount of error.

Proposition 2: The allowable probability distribution for Δ_{ki}, $k = 1, 2, \cdots$, and $i = 1, 2, \cdots, p$, maximizing the probability criterion, is necessarily a symmetric Bernoulli distribution.

Proof: From Proposition 1, recall that $\xi^2 > 1/\rho^2$ where ρ^2 and ξ^2 correspond to a distribution P that contains more than two support points. It then follows that $\|(1/\rho)^2 d\| < \|\xi^2 d\|$ where the left-hand side of the inequality is equal to the bias term under $\Delta_{ki} = \pm\rho^{-1}$ [see (2)]. Now, according to (2), the covariance terms under P and under the distribution $\Delta_{ki} = \pm\rho^{-1}$ are both given by $\rho^2 D$. Hence $\mathrm{Pr}(Z \in V_\theta)$ is larger under the Bernoulli distribution, and the necessity follows. Note that the result holds *regardless* of the values of d and D. $\square$

Numerical procedures for optimizing J, given an implicit *a priori* model for the loss function, are similar to the procedure described in Remark 1; they involve the application of Bernoulli-distributed perturbation sequences and numerical assessment of $\mathrm{Pr}(Z \in V_\theta)$.

Remark 2: Consider the degenerate case $d = 0$. This, for example, occurs when the third-order derivatives of the loss function at θ^* are zero; see [8]. Then, clearly, the optimal solution according to both the mean square error and probability criteria will be a distribution with $\rho \to 0$, forcing the covariance $\rho^2 D$ to zero. This implies that $\Delta_{ki} \to \pm\infty$ is the optimal choice for random perturbations. However, $\lim_{k\to\infty} c_k = 0$, meaning that it is not possible to draw any definitive conclusion about the optimal size of $c_k\Delta_k$ based on the asymptotic properties. In finite sample cases, c_k does not get infinitesimally small, and it is obviously not allowed to let $|\Delta_{ki}| \to \infty$, either. However, a practical guideline in $d = 0$ situations is to select the magnitude of Δ_{ki} as large as the algorithm does not go unstable. This example shows that the asymptotic results must be interpreted and used with some care in finite sample cases.

IV. NUMERICAL STUDY

In this section, we apply SPSA to a statistical experiment design problem for parameter estimation in a dynamic model; see, e.g., [14]. Consider the following autoregressive model with exogenous inputs [ARX(2, 1)]:

$$y_t = h_1 y_{t-1} + h_2 y_{t-2} + u_t + e_t \qquad (6)$$

where $\{u_t\}$ and $\{y_t\}$ are input and output sequences and $\{e_t\}$ is a sequence of mean zero i.i.d. Gaussian random variables. We assume that the input sequence is generated by a *finite* register with length 10 (i.e., the input repeats periodically with cycle 10). We wish to compute the input sequence parameter $(u_1, \cdots, u_{10})^\top$ which starting from zero initial condition minimizes

$$J_u = -E\{\log \det M_F\} + 0.5 \sum u_t^2 \qquad (7)$$

where

$$M_F = \begin{bmatrix} \sum_{i=n_1}^{n_2} y_{t-1}^2 & \sum_{i=n_1}^{n_2} y_{t-1} y_{t-2} \\ \sum_{i=n_1}^{n_2} y_{t-1} y_{t-2} & \sum_{i=n_1}^{n_2} y_{t-1}^2 \end{bmatrix}.$$

Notice that such a problem formulation implies that we deal with a static optimization problem and not a dynamic one since we consider the whole sequence of data $\{y_t\}$ in batch mode within the loss function and a fixed number of parameters independent of the size of the data set. We explain (7) as follows. Assuming that we are interested in estimating $\Lambda = (h_1, h_2)^\top$, the basic least squares estimate is given by (see, e.g., [10])

$$\hat{\Lambda} = M_F^{-1} \begin{bmatrix} \sum_{i=n_1}^{n_2} (y_t - u_t) y_{t-1} \\ \sum_{i=n_1}^{n_2} (y_t - u_t) y_{t-2} \end{bmatrix}.$$

Hence, by selecting the input sequence to maximize the expected value of the (logarithm) of the determinant of M_F, we wish to avoid the problem of the singularity of M_F. Indeed, for large values of sample size, the matrix M_F is (approximately) proportional to Fisher's information matrix for the model given by (6) (see [14, ch. 6]). This choice of optimality criterion is called D-optimality in the statistical experiment design literature; see, e.g., [16]. Since the positive semidefinite matrix M_F is an increasing function of the input power $\sum u_t^2$, the second term of the criterion penalizes signals with large power. For a detailed treatment of the problem of input design for dynamic system identification, see [14, ch. 6]. The optimal design is usually obtained by assuming a model for the data and calculation of the information matrix as a function of input. Such models are often obtained through performing preliminary identification experiments. Here, we directly estimate the optimal inputs without requiring a preliminary identification stage.

Let us assume that the model parameters are given by $h_1 = 1.45$, $h_2 = -0.475$ (which correspond to poles 0.5 and 0.95), and the standard deviation of e_t is 0.05. Note that these values are used for data generation purposes and to (approximately) determine the optimal distribution of the random perturbations. The SPSA algorithm requires no knowledge of these values and the optimization may be carried out by real experimentations that involve exciting the system at initial rest by different inputs and output measurements to compute J_u. In the following, we select $n_1 = 9$, $n_2 = 64$, $a_k = 0.1/k^{0.9}$, and $c_k = 1/k^{0.15}$.

We first apply SPSA with 50 000 iterations, $q = 1$, and $\Delta_{ki} = \pm 0.1$ (Bernoulli distributed) in order to obtain an estimate of the (uncomputable) optimal sequence $\{u_t^*\}$ for later reference. This value will be used as the true optimum for the rest of the paper since the number of iterations for all later estimation is $1200 \ll 50\,000$. Then, we assess the second- and third-order derivatives of the loss function at the optimum, $\{u_t^*\}$, by numerical finite difference method for the noise free case. Also, we approximate σ^2 by simulation of 1000 realizations of $[\log \det(M_F)]$ at $\{u_t^*\}$. Inserting these estimates in (4), we obtain the Bernoulli distribution $\Delta_{ki} = \pm 0.19$. This distribution shall only be used as an initial guess for a numerical search to find the optimizer for the mean square error and probability criteria since only rough estimates of K_1 and K_2 [see (4)] are available. We apply the numerical procedure described in Remark 1 and the comment below Proposition 2 for Bernoulli-distributed random perturbations with varying magnitudes (around 0.19). Using 100 cross sections for each distribution, we obtain Table I where the top row provides the magnitude of the applied Bernoulli distribution. For the probability

TABLE I
PERFORMANCE OF SPSA UNDER VARYING BERNOULLI DISTRIBUTIONS

	± 0.15	± 0.25	± 0.4	± 1
MSE	0.0063	0.0052	0.0073	0.1061
J	0.36	0.51	0.35	0.0

criterion, we have chosen the special case below (5) with $A = 4 \times 10^{-3}$. The results indicate that an inappropriate choice of random perturbations (e.g., ± 1 in this numerical study) would lead to very poor estimation properties.

We also apply a random variable uniformly distributed over $[-0.3, -0.2] \cup [0.2, 0.3]$. This choice is interesting since the support of the distribution includes the optimal support point of the optimal Bernoulli (± 0.25). The numerical evaluations of MSE and J yield 0.0062 and 0.39, respectively, which are noticeably worse than the results for the optimal Bernoulli distribution.

Finally, in order to investigate the performance of the asymptotic solution for small sample cases and large initial deviations from the true optimum, consider a case of 10 iterations with a 17.5% initial deviation for all components of $\{u_t\}$. We are particularly interested in numerically determining whether or not a *distribution form* other than symmetric Bernoulli yields better results than the asymptotically optimal solution. Therefore, we test the optimal Bernoulli (± 0.25) distribution against two bimodal distributions. One is chosen to be a random variable uniformly distributed over $[-0.3, -0.2] \cup [0.2, 0.3]$. The other corresponds to a random variable triangular distributed over both $[0.2, 0.3]$ and $[-0.3, -0.2]$. The corresponding MSE values are 0.0756, 0.0789, and 0.0764, respectively. This comparison obviously does not establish the optimality of the Bernoulli distribution in all finite sample cases. However, it indicates that the asymptotic solution may perform reasonably well even for very small sample sizes.

V. CONCLUDING REMARKS

The paper deals with the optimal choice of random perturbations for the SPSA algorithm. Since the user has full control over this choice, there is strong reason to pick this distribution wisely in order to reduce the overall costs of optimization. We have shown that for the mean square error and probability criteria, the optimal random perturbations should be necessarily sampled from a symmetric Bernoulli distribution. While the optimal Bernoulli distribution (i.e., the magnitude of its outcome) is dependent upon the prior information about the loss function, the optimality of the Bernoulli *distribution form* holds *regardless* of the prior information. This has significant practical implications as the perturbation distribution is typically determined based on small scale experimentation and/or limited prior knowledge about the form of the loss function. All the results are based on the asymptotic theory. Investigating the choice of random perturbations for finite sample cases is of significant theoretical and practical interest and represents a possible topic for future research on the subject.

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Robust Stabilization of Uncertain Systems with Time-Varying Multistate Delay

Yong-Yan Cao and You-Xian Sun

Abstract—In this paper, the authors deal with the problem of stabilizing a class of uncertain linear systems with time-varying multi-state delay and subject to norm-bounded parameter uncertainty via memoryless linear state feedback. Some sufficient conditions for the robust stabilizability are derived for this class of uncertain systems. If there exists a positive-definite symmetric solution satisfying the algebraic Riccati equation (or inequality), a suitable memoryless state feedback law can be derived also. Moreover, all such parametric algebraic Riccati inequalities have been transformed into some linear matrix inequality problems, so there is no tuning of the parameters to gain a stabilizing solution.

Index Terms— Algebraic Riccati Equation (ARE), linear matrix inequality (LMI), linear system, time-delay, uncertainty.

I. INTRODUCTION

Dynamical systems with time-delay are common in chemical processes and long transmission lines in pneumatic, hydraulic, and rolling mill systems. A major problem in the analysis of linear dynamical systems with time-delay is related to the stability and the stabilization using linear feedback with or without memory. Several results are readily available in the literature; see, e.g., [1]–[3] and the

references cited therein. Some of the results have been successfully extended to include the effect of bounded uncertainties [4]–[11].

Recently, some researchers have proposed some useful techniques to determine a linear control law which quadratically stabilizes a linear system with norm-bounded time-varying uncertainties [9]–[11]. They considered systems with time-varying uncertainties and obtained the constant feedback gains for linear stabilizing controllers in terms of the solution of a parametric Riccati equation. Furthermore, these results have been generalized to linear systems with a single state-delay and norm-bounded time-varying uncertainties in [5] and [6]. But those theories presented can be applied only to systems which satisfy the so-called matching condition, which is known to be restrictive in general applications; e.g., see [9]. The research in [7] and [8] has focused on the use of rank-one decomposition of uncertainties. But in [7], the model has ignored the nominal (certain) effects of the delayed state and the decomposition of the norm-bounded uncertainties is quite restrictive. In [8], the feedback control law cannot be gained often since the ARE is too conservative because the same one parameter has been used to manipulate the uncertainties in state, delayed state, and input.

The major contributions of this work are divided into four parts. First, it gives a new criterion of the delay-independent stabilizability in the form of the algebraic Riccati inequality (ARI) and linear matrix inequality (LMI) for linear uncertain systems with time-varying state delays. Second, it treats directly the systems with multi-state delay. Third, it extends the results from [9] and [10] to a class of uncertain dynamic time-delay systems and obtains the memoryless linear state feedback control law which renders the closed-loop system asymptotically stable for all realizations of uncertainty in terms of the solution of a parametric Riccati equation. Fourth, an LMI approach is proposed for solving the above robust control problems. This approach has the advantage that no tuning of parameters and/or positive definite matrices is involved, as in the case with the robust stabilization methods of [5]–[11], and it can be computed very efficiently [12].

Throughout this paper, we let $\mathbf{R} = (-\infty, +\infty)$, $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R}^n$ be any real n-dimensional linear vector space over the real equipped by the standard Euclidean norm $\| \cdot \|$. The matrix I denotes an identity matrix. We use $W > 0 (<0)$ to denote a positive-definite (negative-definite) symmetric matrix W.

II. SYSTEM MODEL AND DEFINITION

Consider the uncertain time-delay system described by the following functional differential equation:

$$\dot{x}(t) = [A_0 + \Delta A_0(r_0(t))]x(t) + \sum_{i=1}^{q} [A_i + \Delta A_i(r_i(t))] \\ \cdot x(t - h_i(t)) + [B + \Delta B(r_{q+1}(t))]u(t) \quad (1)$$

$$x(t) = \phi(t), \forall t \in [-h_{\max}, 0) \quad (2)$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^n$ is the control input, and $A_i = 0, 1, \cdots, q$ and B are known real constant matrices of appropriate dimensions which describe the nominal system. The matrices $\Delta A_i(\cdot)$, $i = 0, 1, \cdots, q$ and $\Delta B(\cdot)$ are real-valued functions representing time-varying parameter uncertainties, and $h_i(t)$, $i = 1, \cdots, q$ are time-varying bounded delay times satisfying

$$0 < h_i(t) \le h_i < \infty, \dot{h}_i(t) \le d_i < 1$$
$$h_{\max} = \max_i(h_i), d_{\max} = \max_i(d_i). \quad (3)$$