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An Eigenvalue Study of a Double Integrator Oscillator

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Abstract—A tutorial study of an oscillator built from a loop of two active \( RC \) integrators and an ideal inverter. As a reference the linear harmonic oscillator is modeled as a \( LC \) circuit and as an ideal two integrator loop. The nonlinear amplifiers of the active \( RC \) integrator circuits are assumed to be linear with time varying gain and the eigenvalues are found as function of time. A design strategy based on the time-constants of the integrators is presented.

I. INTRODUCTION

In the recent 20 years the classic circuit theory with analysis and synthesis of linear electrical circuits has more or less disappeared from the curriculum of the electrical and electronic engineering students. The digital systems dominate over the analog systems. But in the future possibly nonlinear analog systems will be used instead of or as a supplement to digital systems due to lower power consumption and higher speed [1]. The aim of this tutorial is to demonstrate that the classic circuit theory with poles and zeros (eigenvalues) may be used to gain insight in the behavior of nonlinear circuits.

![Harmonic Oscillator Models](image)

**Fig. 1.** Harmonic Oscillator Models.

\[ s = \pm j \frac{1}{\sqrt{L+C}} \]

\[ s = \pm j \frac{1}{\sqrt{C_1+C_2}} \]

\[ x = V(3) = V(1) = \cos(t), \quad y = I(L) = V(2) = \sin(t) \]

![Comparison of Harmonic Oscillator Models](image)

**Fig. 2.** Comparison of Harmonic Oscillator Models. FFT analysis. \( V(1) = V(3) = \cos(t), \quad V(2) = \sin(t) \)

II. HARMONIC OSCILLATOR MODELS

The harmonic oscillator may be modeled as a capacitor coupled in parallel to a coil, Fig. 1(a). If the initial condition is a voltage across the capacitor and a current equal to zero through the coil the voltage becomes a cosine of time and the current becomes a sine of time. The capacitor may be modeled as a current source controlled by the time derivative of its voltage. The coil may be modeled as a voltage source controlled by the time derivative of its current. As variables we choose the voltage across the capacitor: \( x \), and the current through the coil: \( y \), (state variables). We choose voltage \( V \) and current \( I \) as variables because they are easy to measure (signals). We should use charge \( q = C \cdot V \) and flux \( \phi = L \cdot I \) as variables because they represent the energy in the system. Unfortunately they are difficult to measure. In the real world charge and flux are nonlinear functions of voltage and current. Note that current is time derivative of charge \( I = dq/dt \) and voltage is time derivative of flux \( V = d\phi/dt \). Charge provide the coupling between the chemical world and the electrical world (electrolysis). Flux provide the coupling between the mechanical world and the electrical world (generators, motors). Capacitors and coils are memory elements. The resistor
with memory - the memristor $\phi = M \ast q$ - has recently been implemented in the real world [2].

\[ \phi = M \ast q \]

now the harmonic oscillator may be modeled (defined) by means of two first order differential equations

\[ L \frac{dy}{dt} = +x \]

\[ C \frac{dx}{dt} = -y \]

The two equations may be combined into a second order differential equation.

\[ \frac{d^2x}{dt^2} + \frac{x}{LC} = 0 \]

The root of the characteristic polynomial - the eigenvalues - are a complex pole pair $s = \pm j \omega$ on the imaginary axis where $\omega = 2\pi f$. With $\omega^2 = 1/(LC)$ the frequency becomes $f = 1/(2\pi \sqrt{LC})$. The amplitude is given by the initial condition.

An ideal integrator may be modeled as a capacitor loaded current source. The voltage of the capacitor is the time integral of the current source. By means of two ideal integrators the harmonic oscillator may be modeled as shown in Fig. 1(b).

The figures Fig. 2 and Fig. 3 show a PSpice comparison of two 100kHz oscillators with the following component values: $L_3 = 1.591549431\mu H$, $C_3 = 1.591549431\mu F$ and $C_1 = C_2 = 1.591549431\mu F$. With an accuracy of $RELTOL = 1\mu = 1e - 6$ in PSpice a very close agreement between the models is observed.

Linear steady state oscillators are mathematical fiction. In the real world linear systems are always damped due to losses. Some kind of non-linearity must be introduced in order to obtain steady state oscillations. Oscillators are non-linear systems. It is to be expected that oscillators based on a double integrator will behave very close to sinusoidal so they have been reported frequently in the literature [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. In the following a double integrator oscillator based on operational amplifiers is investigated.

The circuit is a dedicated analogue computer circuit based on the definition of the sine and cosine functions: $d(sin(t))/dt = cos(t)$ and $d(cos(t))/dt = -sin(t)$. Input to the first integrator is $-sin(t)$ which give rise to input $cos(t)$ of the second integrator with output $sin(t)$ to be inverted for feedback to the first integrator (see Fig. 4). Assuming ideal operational amplifiers ($A_1 = A_2 = \infty$) the relation between the components and the frequency becomes $\omega_0^2 = (2\pi f_0)^2 = 1/(\tau_1 \ast \tau_2)$ where $\tau_1 = R_1 \ast C_1$ and $\tau_2 = R_2 \ast C_2$ are the time constants of the integrators. If we choose $R_1 = R_2 = R = 10k\Omega$ and $C_1 = C_2 = C = 15.91549431\mu F$ the nominal oscillating frequency becomes $f_0 = 1\ kHz$, $\tau_1 = \tau_2 = \tau = 159.1549431e - 6$, $\omega_0 = 1/(RC) = 6.283185308e + 3 = 2\pi f_0$.

The two ideal operational amplifiers are replaced with PSpice op-amp macro-models, $\mu A741$ and an ideal PSpice inverter model (EVA3 1 0 5 0 -1) is introduced. The power supply is $\pm 22$ volts.

Figure 5 shows the frequency spectrum of the oscillator in steady state (FFT analysis over 800ms). Higher harmonics may be observed. In this case - where the resistors $R_1$ and $R_2$ are chosen equal and the capacitors $C_1$ and $C_2$ are chosen equal...
- clipping of both amplifier output voltages $V(3)$ and $V(5)$ is observed, Fig. 6.

Figure 6 shows that energy is transferred to the amplifiers as very narrow pulses at the maximums of the output voltages. This behavior is very similar to the behavior of the pendulum clock where the escape mechanism [13] delivers the energy as pulses when the angel from vertical is zero and the weights go down a step changing potential energy into a kinetic energy impulse. For small swing - i.e. when $x$ and $\sin(x)$ are almost equal - the pendulum clock is very close to a damped linear oscillator with very high quality factor and no harmonics. A strategy for design of electronic oscillators with minimum distortion could be optimization of the energy impulses observed.

A large number of PSpice simulations have been made with various combinations of the values of the two resistors and the two capacitors. Apparently it is possible to remove the energy impulses in connection with one of the amplifiers so that the harmonics in the output voltage becomes smaller. The following PSpice results demonstrates this assertion and gives more insight in the behavior of the oscillator.

By means of the following values a number of PSpice simulations have been made: $R_1 = 8.33333333\Omega$, $C_1 = 10.0nF$, $R_2 = 12.0k\Omega$, $C_2 = 25.33172484nF$, time constant amplifier $A_1$: $\tau_1 = R_1 \times C_1 = 83.33333333e-6$, time constant amplifier $A_2$: $\tau_2 = R_2 \times C_2 = 303.9806938e-6$, $\omega^2 = 1/(\tau_1 \times \tau_2) = 39.47619124e-6 = (2\pi f)$, $f = 0.9999718022kHz$.

A comparison of Fig. 7 with Fig. 5 shows that the harmonics of the output voltage $V(5)$ of amplifier $A_2$ have been reduced. Figure 8 shows that the power is supplied to amplifier $A_1$ with an almost constant current of $2.44mA$ and pulses of $15\mu A$ in the short clipping time intervals. The power is supplied to amplifier $A_2$ with an almost constant current of $2.438mA$. A comparison of Fig. 8 with Fig. 6 shows that the pulses in the power supply currents of amplifier $A_2$ have disappeared. This result depends of course on the op-amp macro-model used. Experiments with LM741 instead of uA741 did not show pulses but reduction of harmonics was obtained. Figure 9 shows the amplifier gains as functions of time over two periods. It is seen that the gain is varying slowly in the interval $\pm 5000$ in the whole period except at the maximums of the amplifier output voltages where it vary between very large and very small values in very short time intervals.

B. Eigenvalue calculation

A study of the eigenvalues of the time-varying Jacobian of the linearized differential equations may give some insight in the behavior of the oscillator [14]. The linear time-varying approach (LTV) is a method to to calculate the time-varying
eigenvalues (dynamic eigenvalues) of a nonlinear circuit [15], [16], [17], [18], [19]. To calculate the dynamic eigenvalues, the Riccati equation must be solved. In the following it is assumed that a nonlinear circuit can be treated as a time-varying linear circuit.

If it is assumed that the amplifiers are perfect - i.e. the input impedance is infinite and the output voltage is equal to the input voltage times a time varying constant $A$ i.e. $V_{out} = (V_+ - V_-) * A(time)$ - the characteristic polynomial of the circuit becomes

$$s^2 + 2\alpha s + \omega^2 = 0 \quad (4)$$

where

$$2\alpha = \frac{C_1 R_1 (1 + A_1) + C_2 R_2 (1 + A_2)}{C_1 R_1 C_2 R_2 (1 + A_1)(1 + A_2)}$$

and

$$\omega^2 = \frac{1 + A_1 A_2}{C_1 R_1 C_2 R_2 (1 + A_1)(1 + A_2)}$$

The roots are

$$p_{1,2} = -\alpha \pm j\sqrt{\omega^2 - \alpha^2}$$

The roots of the characteristic polynomial (the poles) are found as function of time by means of a table of the gains as function of time found by means of PSpice. The result is shown in Fig. 10. It is seen that the imaginary part is close to $\omega_0$ over a period. The time constants of the integrators are chosen different in order to obtain minimum distortion of the output voltage of one of the amplifiers. The circuit is used mainly for low-frequency oscillators. The structure may be a candidate for IC implementation because it is based on resistors and capacitors only.

IV. CONCLUSION

It is demonstrated that the classic circuit theory with poles and zeros (eigenvalues) may be used in connection with the design of a double integrator oscillator. The non-linear circuit is treated as a linear time-varying circuit. The parameters are chosen in such a way that the imaginary part of the poles of the linear time-varying circuit is as close as possible to $\omega_0$ over a period. The time constants of the integrators are chosen different in order to obtain minimum distortion of the output voltage of one of the amplifiers. The circuit is used mainly for low-frequency oscillators. The structure may be a candidate for IC implementation because it is based on resistors and capacitors only.

REFERENCES

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Abstract
A tutorial study of an oscillator built from a loop of two active RC integrators and an ideal inverter. As reference the linear harmonic oscillator is modeled as a LC circuit and as an ideal two integrator loop. The nonlinear amplifiers of the active RC integrator circuits are assumed to be linear with time varying gain and the eigenvalues are found as function of time. A design strategy based on the time-constants of the integrators is presented.

The aim of this tutorial is to demonstrate that the classic circuit theory with poles and zeros (eigenvalues) may be used to gain insight in the behavior of nonlinear circuits

COMPUTER AIDED CIRCUIT ANALYSIS
- The kernel of analyzing nonlinear circuits is the solution of a linear circuit
- All elements may be modelled by means of a controlled source
- Obtained obtained equations may be approximated with either a dynamic value or a static value

The CAPACITOR is a voltage controlled current source (VCCS) controlled by the time derivative of its voltage: IC = dQ/DT = C dV/DT

The TORQUE is a current controlled voltage source (CCVS) controlled by the time derivative of its current: VL = dI/L = L dI/DT

EXTENDED NODE EQUATIONS
- First order differential equations
- Variables: Node Potentials and Impedance Branch Currents
- Current sources: Admittance Branches
- Voltage sources: Impedance Branches

"FREEZE" TIME and replace nonlinearity with dynamic ratio (Jacobian) or with static ratio (LTV)

HARMONIC OSCILLATOR MODELS
Passive lossless LC circuit
Active ideal integrator circuit

FREQUENCY RESPONSE ANALYSIS
- By means of MEASUREMENTS you may investigate the behaviour of the REAL WORLD
- By means of SIMULATION you may investigate the behaviour of your MODEL

If you achieve AGREEMENT between measurements and simulation YOU BELIEVE IN YOUR MODEL
- By means of a POSTER you may present the results of a LECTURE of several hours

NUCLEAR INTEGRATION with Variable Integration Step and Variable Order Polynomial Approximation
Newton-Raphson Iteration Replace with dynamic value
Pascal Iteration Replace with static value

G=L/I

Nonlinear controlled source i = G(v)

Numerical Amplifiers are made from transistors, resistors and capacitors - i.e. they are high order nonlinear circuits

Operations are based on either circuit element modeling (e.g. BJTs, transistors, resistors and capacitors) or macromodeling (controlled sources, diodes, resistors, capacitors)
It makes sense to study the eigenvalues of the linear circuit signal models used in each integration step of the numerical simulation process

Physical comparison of two 10 kHz oscillators with the following component values:
L1 = 1.59x10^-4 H
C1 = 1.59x10^-6 F
and
C1 = 2R = 1.59x10^-6 F

With an accuracy of BEETLE = t = 1.0 x 10^-3 in Picture 1 very close agreement between the models is observed

The initial condition is a voltage across the capacitor and a current equal to zero through the coil.

The voltage becomes a cosine of time and the current becomes a sine of time

The characteristic polynomial of the circuit becomes

\[ s^2 + 2\alpha s + \omega_0^2 = 0 \]

where

\[ \alpha = \frac{1}{2} \left( \frac{1}{L} + \frac{1}{R} \right) \]

The roots are

\[ \alpha = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1}{L} + \frac{1}{R}} \]

Energy is transferred to the amplifier in a very narrow pulse at the maximum of the output voltage.

The behavior is very similar to the behavior of the periodic clock where the input nonlinearity dictates the energy in the pulse when the stage from normal to zero and the output amplifier acts as a switch. The voltage is transferred to the amplifier at the point where the output voltage has the value of the voltage across the amplifier and the internal equal - the periodic clock is very close to a drop out oscillator with very high quality factor and a very low Q factor.

A strategy for design of electronic oscillators with pull-in : Stability and optimization of the energy transfer is obtained.

COMPLEX POLE PAIR as function of time

\[ s = \frac{1}{2} \left( \frac{1}{L} + \frac{1}{R} \right) \pm \frac{1}{2} \sqrt{\frac{1}{L} + \frac{1}{R}} \]

\[ s = \frac{1}{2} \left( \frac{1}{L} + \frac{1}{R} \right) \pm j \frac{1}{2} \sqrt{\frac{1}{L} + \frac{1}{R}} \]

The complex pole pair is a function of time.

The complex pole pair will be close to the real poles at the right side of the complex plane.

The result is in agreement with the very low phase noise

CONCLUSION
A tutorial technique for designing nonlinear oscillators is presented with the use of real world measurements.

The two non-linear circuits are modeled as linear circuits. The design strategy is based on the time-constant of the integrators. The linear oscillator is modeled as an ideal two integrator loop. The nonlinear amplifiers of the active RC integrator circuits are assumed to be linear with time varying gain and the eigenvalues are found as function of time. A design strategy based on the time-constants of the integrators is presented.