



## Synthesis of nonuniformly spaced arrays using a general nonlinear minimax optimisation method

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$$F_z = C_1 \iint \sqrt{G_f(\theta, \phi)} \sin \frac{\theta}{2} \frac{(\cos \phi e_{rx} + \sin \phi e_{ry})}{\rho} \cdot \exp[-jk\rho(1 - \bar{u}\rho \cdot \bar{u}R)] dS \quad (31c)$$

in which  $C_1 = (jk\eta/4)C(e^{-jkR}/R)$  and the integration is to be performed on the paraboloidal surface with respect to  $(\rho, \theta, \phi)$ .

It is interesting to note that (31) and Table I are the same except for a sign factor. This shows that the radiation from the aperture plane is that of a dipole instead a Huygens source. This comes from the fact that surface current distribution (30) is calculated from  $\bar{H}_t$  only. So the far-field calculations correspond to (19). In fact, there is a constant proportionality between  $f_x$ ,  $F_x$ , and  $f_y$ ,  $F_y$  from (19) and (31). As  $D/\lambda$  increases, the angle  $\Theta$ , which sees the main lobe and first few sidelobes, gets smaller so the term  $(1 - \cos \Theta)$  approaches zero yielding zero far-field cross polarization provided polarization purity is achieved at the aperture plane.

### CONCLUSIONS

All these considerations lead us to the conclusion that the cross polarization performance of reflector antenna system largely depends on the feed system and  $\lambda/D$  ratio. For this reason, superior performance of Cassegrainian or front-fed paraboloids with respect to polarization purity is closely related to the feed design. Thus this paper puts an end to the controversy recently arisen ([7]–[9]) about the immunity of these two types of antenna systems against depolarization effects. A detailed treatment may be required to see to what extent other factors may influence the cross polarization performance of reflector antennas.

### REFERENCES

- [1] A. C. Ludwig, "The definition of cross-polarisation," *IEEE Trans. Antennas Propagat.*, vol. AP-21, pp. 116–119, January 1973.
- [2] Collin and Zucker, *Antenna Theory*. New York: McGraw-Hill, 1969, Part 1, pp. 62–74.
- [3] Watson and Ghobrial, "Off-axis polarisation characteristics of Cassegrainian and front-fed paraboloidal antennas," *IEEE Trans. Antennas Propagat.*, vol. AP-20, pp. 691–698, 1972.
- [4] S. Silver, *Microwave Antenna Theory and Design*. M.I.T. Radiation Laboratory, 1964, p. 335.
- [5] J. H. Cowan, "Dual-band reflector-feed element for frequency-reuse applications," *Electronics Letters*, vol. 9, no. 25, pp. 596–597, 13th December 1973.
- [6] Collin and Zucker, *Antenna Theory*. New York: McGraw-Hill, 1969, Part 2, pp. 41–48.
- [7] P. J. Wood, "Depolarisation with Cassegrainian and front-fed reflectors," *Electronics Letters*, vol. 3, pp. 181–183, 3rd May 1973.
- [8] Watson and Ghobrial, "Cross-polarisation in Cassegrain and front-fed antennas," *Electronics Letters*, vol. 9, no. 14, pp. 297–298, 12th July 1973.
- [9] P. J. Wood, "Cross-polarisation with Cassegrainian and front-fed reflectors," *Electronics Letters*, vol. 3, no. 25, pp. 597–598, 13th December 1973.

## Synthesis of Nonuniformly Spaced Arrays Using a General Nonlinear Minimax Optimization Method

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**Abstract**—Antenna patterns can be synthesized using a new nonlinear minimax optimization method with sure convergence properties. Not requiring derivatives, the proposed method is general and easy to use

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so that it might be applied to a wide variety of nonlinear synthesis problems for which analytical solutions are not known. To test the algorithm a group of test problems for which exact analytical solutions are known has been considered, namely, optimization of Dolph-Chebyshev arrays by spacing variation. The method is further applied to find the element positions in nonuniformly spaced linear arrays with uniform excitation that produce minimized (equal) sidelobe levels, and comparisons are made with conventional Dolph-Chebyshev arrays.

### I. INTRODUCTION

The minimax problem of minimizing the maximum sidelobe level in the pattern with prescribed beamwidth from a linear half-wavelength spaced array was solved analytically by Dolph [1] by identifying the array factor with Chebyshev polynomials. The present paper is concerned with minimax synthesis of antenna patterns in cases that do not have known analytical solutions. Hence, we will primarily be dealing with iterative methods.

Ma [2] described the application of perturbational and linear programming techniques to nonuniformly spaced arrays. James [3] found the excitation in arrays with fixed spacings by linear programming and further introduced nonlinear constraints to maintain a specified 3 dB beamwidth while minimizing the maximum sidelobes. Recently Streit [4] derived sufficient conditions that optimum beam patterns exist for unequally spaced arrays, and he gave examples of determination of optimum element currents by means of the Remez exchange algorithm, however, with fixed spacings. To the authors' knowledge no examples of general minimax optimization of antenna patterns in cases where the approximation errors are nonlinear functions of the design parameters have appeared in the literature. It seems, therefore, well motivated to try to develop and apply efficient nonlinear minimax optimization algorithms.

In this paper we describe a recently developed algorithm for nonlinear optimization using minimax objectives. The algorithm does not require evaluation of derivatives and hence it may be applied to design more complex antenna systems where derivatives are not easily obtained by analytical expressions. The application of the proposed algorithm is demonstrated by simple, yet typical, nonlinear minimax problems: minimum sidelobe synthesis of arrays by variation of element spacings. First we give a formulation of the general problem and describe the algorithm.

### II. STATEMENT OF GENERAL SYNTHESIS PROBLEM

Let the desired antenna pattern  $P_D$  to be synthesized be a function of some field coordinates  $\psi$

$$P_D = P_D(\psi) \quad \psi = \{\psi_1, \psi_2, \psi_3\}. \quad (1)^1$$

The design parameters to be varied are elements in an  $n$ -dimensional vector  $x$  and the actual pattern obtained is denoted

$$P = P(x, \psi). \quad (2)$$

Given a set of  $m$  field coordinates  $\psi_1, \psi_2, \dots, \psi_m$ , we then define an  $m$ -dimensional vector of residuals  $f(x)$  with the components

$$f_j(x) \equiv w_j [P(x, \psi_j) - P_D(\psi_j)], \quad j = 1, \dots, m, m \geq n. \quad (3)$$

The residuals are measures of the errors by which the actual pattern approximates the desired pattern at the  $m$  sample points under consideration. The  $w_j$  are weights that are normally chosen equal to unity. However, if relatively low errors are desired in

<sup>1</sup> As an example,  $\psi$  may be defined by spherical coordinates:  $\psi = \{r, \theta, \phi\}$ .

certain regions the corresponding weights should be chosen greater than 1.

Our problem is now to minimize with respect to  $\mathbf{x}$  the maximum error  $F(\mathbf{x})$  defined by

$$F(\mathbf{x}) \equiv \|\mathbf{f}(\mathbf{x})\| \equiv \max_{1 \leq j \leq m} |f_j(\mathbf{x})| \quad (4)$$

to achieve the desired minimax solution to the pattern synthesis problem.

It should be pointed out that we consider the general case where the residuals are nonlinear functions of the design parameters. Note also that no restriction has been put on the type of patterns considered; both real- and complex-valued patterns are covered.

In the next section we intend to minimize (4) by developing an iterative technique with sure convergence properties.

### III. NONLINEAR MINIMAX OPTIMIZATION

The iterative method for minimization of (4) is based on successive linear approximations to the nonlinear residuals (3). At the  $k$ th stage of the iteration let  $F_k(\mathbf{h})$  be defined through

$$F_k(\mathbf{h}) \equiv \|\mathbf{f}(\mathbf{x}_k) + \mathcal{B}_k \mathbf{h}\| \quad (5)$$

where  $\mathcal{B}_k$  is an approximation to the derivative matrix  $\{\partial f_j / \partial x_i\}$  found by means of the Broyden updating formula [5] without extra calculations of the residuals. Thus, for small values of  $\|\mathbf{h}\|$ ,  $F_k(\mathbf{h})$  is an approximation to  $F(\mathbf{x}_k + \mathbf{h})$ . The increment  $\mathbf{h}_k$  to the current point  $\mathbf{x}_k$  is then found by solving the constrained linear minimax problem

$$F_k(\mathbf{h}_k) = \min_{\|\mathbf{h}\| \leq \lambda_k} [F_k(\mathbf{h})] \quad (6)$$

which may be set up as a linear programming problem and solved by any suitable linear programming routine. However, for greater efficiency we use a method [6] similar to the exchange algorithm for solving systems of linear equations in the minimax sense. The point  $(\mathbf{x}_k + \mathbf{h}_k)$  is accepted as the next current point if the decrease in the objective function  $F$  exceeds a small multiple of the decrease in the linear approximation, more precisely if

$$F(\mathbf{x}_k) - F(\mathbf{x}_k + \mathbf{h}_k) \geq \alpha [F(\mathbf{x}_k) - F_k(\mathbf{h}_k)] \quad (7)$$

$\alpha = 0.01$ . Otherwise we let  $\mathbf{x}_{k+1} = \mathbf{x}_k$ .

The bound  $\lambda_k$  is adjusted during the iterations according to the following strategy. If the decrease in the objective function  $F$  is poor compared to the decrease predicted by the linear approximation  $F_k$ , then the step length may be too large. Consequently, if (7) is not satisfied with  $\alpha = 0.1$ , we let  $\lambda_{k+1} = 0.7\|\mathbf{h}_k\|$ . Alternatively the step length may be unnecessarily small if the agreement between each residual  $f_j$  and its linear approximation is very good at the point  $\mathbf{x}_k + \mathbf{h}_k$ . Therefore we test the inequality

$$\|\mathbf{f}(\mathbf{x}_k + \mathbf{h}_k) - [\mathbf{f}(\mathbf{x}_k) + \mathcal{B}_k \mathbf{h}_k]\| \leq \frac{1}{2} [F(\mathbf{x}_k) - F(\mathbf{x}_k + \mathbf{h}_k)] \quad (8)$$

and if it holds we let  $\lambda_{k+1} = 2\|\mathbf{h}_k\|$ . This is sensible since Taylor's formula states that the term on the left-hand side of (8) is of the order  $\|\mathbf{h}_k\|^2$ , whereas the term on the right-hand side is of the order  $\|\mathbf{h}_k\|$ . In all other cases we let  $\lambda_{k+1} = \lambda_k$ . The various constants governing the optimization strategy have been determined through extensive numerical experiments. It was found, however, that the choice of the constants was not critical.

In order that the matrices  $\mathcal{B}_k$  are good approximations to the derivative matrices, it is important that the directions  $\mathbf{h}_k$  satisfy some linear independence conditions. This is ensured by introducing "special iterations" as suggested by Powell [7]. A

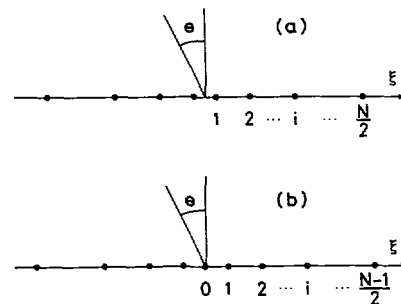


Fig. 1. Unequally spaced symmetric  $N$ -element linear broadside arrays. (a)  $N$  even. (b)  $N$  odd.

description of this is beyond the scope of this paper, details may be found in [7] and [8].

An important feature of the algorithm is that it possesses sure convergence properties.<sup>2</sup> Madsen [9] has shown that if the problem under consideration is nonsingular the final rate of convergence is super-linear. This means that if the optimum solution is denoted by  $\mathbf{x}^*$  the inequality

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq \epsilon_k \|\mathbf{x}_k - \mathbf{x}^*\| \quad (9)$$

where  $\epsilon_k \rightarrow 0$ , will be fulfilled.

A Fortran implementation of the algorithm described above along with another minimax algorithm that requires evaluation of the exact first-order derivatives [10] is available in Madsen and Schjær-Jacobsen [11]. The user has to specify initial values of  $\lambda$  and a general upper bound  $\Lambda$  on the step length. In all examples presented in the following  $\lambda_0 = \Lambda = 0.1$  is used.

### IV. SYNTHESIS OF ARRAYS BY SPACING VARIATION

Consider a linear symmetrical broad-side array with  $N$  elements as an example (Fig. 1). The radiation pattern is given by

$$g_e(\theta) = 2 \sum_{i=1}^{N/2} a_i \cos(2\pi \xi_i u), \quad u = \sin \theta, \quad N \text{ even} \quad (10a)$$

$$g_o(\theta) = 1 + 2 \sum_{i=1}^{(N-1)/2} a_i \cos(2\pi \xi_i u), \quad u = \sin \theta, \quad N \text{ odd} \quad (10b)$$

where  $a_i$  is the (normalized) excitation coefficient and  $\xi_i$  the distance (in wavelengths) from the array center. We now attempt to synthesize arrays with minimum equal sidelobes by varying the element positions rather than the excitation coefficients. This problem clearly represents a nonlinear minimax problem, whereas finding the excitation coefficients for given (unequal) element spacings is a linear problem that may be solved by well-known linear programming techniques.

To minimize the maximum sidelobe level define  $n$  design parameters  $x_i$  such that

$$\xi_i = x_i + \xi_{i-1},$$

$$i = 1, 2, \dots, n \begin{cases} n = \frac{N-2}{2}, & N \text{ even} \\ n = \frac{N-3}{2}, & N \text{ odd} \end{cases} \quad (11a)$$

$$(11b)$$

with  $\xi_0 = 0$  and  $\xi_{n+1} = (\lambda/4)(N-1)$ . Hence,  $x_i$  represents the element spacings, except when  $i = 1$  for  $N$  even in which case  $x_1$

<sup>2</sup> By this term we mean that  $\mathbf{x}_k$  will converge to a stationary point of  $F$ .

is the distance from the array center to element 1. Note that the total length is a fixed constant by this definition.

The residuals in (3) are defined through

$$f_j = g_e(\theta_j)/g_e(0), \quad j = 1, 2, \dots, m \quad (12)$$

where the  $m$  sample points  $\theta_j$  are chosen such that

$$0^\circ < \theta_1 < \theta_2 < \dots < \theta_j < \dots < \theta_m = 90^\circ. \quad (13)$$

This corresponds to the choice  $w_j = 1$  and  $P_D(\theta_j) = 0$  in (3).

V. DOLPH-CHEBYSHEV ARRAYS AS TEST CASES

In order to demonstrate the method on cases where the exact minimax solutions are known, we first consider optimization of Dolph-Chebyshev arrays by variation of element spacings. With the element spacings being  $\lambda/2$ , the normalized excitation coefficients for such arrays may be calculated by the formulas given by Stegen [12]. For six- and eight-element arrays we get

$$(a_1, a_2, a_3) = (1.0, 0.7768, 0.5406), \quad N = 6$$

and

$$(a_1, a_2, a_3, a_4) = (1.0, 0.8751, 0.6603, 0.5799), \quad N = 8$$

corresponding to a sidelobe level of  $1/R = 0.1$  or  $20 \log(1/R) = -20$  dB.

To be able to exactly reproduce the  $\lambda/2$  spacing in an optimization process we have to sample the pattern at the angles  $\theta_p'$  where the optimum pattern attains its maximum sidelobe levels. These angles are calculated by the formula

$$\theta_p' = \frac{180}{\pi} \sin^{-1} \left[ \frac{2}{\pi} \cos^{-1} \left[ \frac{\cos \left( \frac{p\pi}{N-1} \right)}{\cosh \left( \frac{1}{N-1} \cosh^{-1} R \right)} \right] \right],$$

$$\begin{cases} p = 0, 1, \dots, \frac{N}{2}, & N \text{ even} \\ p = 0, 1, \dots, \frac{N+1}{2}, & N \text{ odd.} \end{cases} \quad (14)$$

We also insert additional sample points at essentially  $5^\circ$  spacing so that for  $N = 6$  we sample at the angles  $(\theta_1, \theta_2, \dots, \theta_m)$  given by

$$(21.11, 25.0, 31.43, 35.0, 40.0, \dots, 56.30, \dots, 90.0), \quad m = 15,$$

and for  $N = 8$  at the angles given by

$$(15.32, 20.0, 22.52, 30.0, 37.84, 40.0, 45.0, \dots, 60.40, \dots, 90.0),$$

$$m = 16.$$

In Table I(a) and (b) the number of pattern evaluations required to exceed a specified accuracy  $\delta$  is given for different starting values of the element spacings.<sup>3</sup>  $\delta$  is defined by

$$\delta = \max_{1 \leq i \leq n} \left| \frac{x_i - x_i^*}{x_i} \right| \quad (15)$$

where  $x^* = (0.25, 0.50)$  for  $N = 6$  and  $x^* = (0.25, 0.50, 0.50)$  for  $N = 8$ . From Fig. 2 it is seen that the starting points chosen represent patterns that are considerably different from that of the optimum Dolph-Chebyshev pattern.

<sup>3</sup> These examples were run in double precision Fortran IV on an ICL 1906S computer.

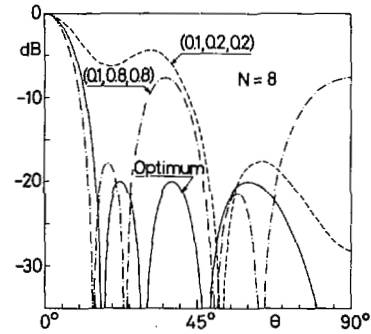


Fig. 2. Optimization of eight-element Dolph-Chebyshev array. Initial patterns at starting points  $(x_1, x_2, x_3) = (0.1, 0.2, 0.2)$  and  $(0.1, 0.8, 0.8)$ . Optimum pattern at solution  $(0.25, 0.5, 0.5)$ .

TABLE I  
OPTIMIZATION OF  $N$ -ELEMENT DOLPH-CHEBYSHEV ARRAYS WITH  $-20$  DB SIDELobe LEVEL

(a) $N = 6$						
Initial values			Number of pattern evaluations			
$x_1$ [ $\lambda$ ]	$x_2$ [ $\lambda$ ]	Max. residual [dB]	$\delta=10^{-2}$	$\delta=10^{-3}$	$\delta=10^{-4}$	
0.1	0.2	-3.743	10	10	11	
0.1	0.8	-3.681	10	11	11	
0.4	0.2	-4.148	8	8	10	
0.4	0.8	-8.678	13	14	16	

(b) $N = 8$							
Initial values				Number of pattern evaluations			
$x_1$ [ $\lambda$ ]	$x_2$ [ $\lambda$ ]	$x_3$ [ $\lambda$ ]	Max. residual [dB]	$\delta=10^{-2}$	$\delta=10^{-3}$	$\delta=10^{-4}$	
0.1	0.2	0.2	-4.408	14	14	15	
0.1	0.2	0.8	-6.362	12	12	14	
0.1	0.8	0.2	-3.630	9	14	17	
a	0.1	0.8	0.8	-7.505	20	21	23
	0.4	0.2	0.2	-2.867	11	14	14
	0.4	0.2	0.8	-3.270	12	14	14
	0.4	0.8	0.2	-8.185	12	15	17
b	0.4	0.8	0.8	-10.436	Convergence to local optimum		

a,b See text for explanation.

A few comments to the entries *a* and *b* in Table I(b) are necessary. Using the sample points previously mentioned the optimization process converged to an improper local optimum for which  $\xi_n$  was larger than  $\xi_{n+1}$ , i.e., the array elements were no longer correctly ordered. It was then decided to use much denser sample points, namely,

$$(15.32, 16.0, 17.0, \dots, 22.0, 22.52, 24.0, \dots, 37.0, 37.84, 39.0, \dots,$$

$$59.0, 60.40, 61.0, \dots, 89.0, 90.0), \quad m = 76$$

whereupon the correct optimum was obtained with starting point *a* as stated in Table I(b) while starting point *b* still produced the local optimum. Starting point *b* is characterized through an initially incorrect ordering of the elements and is thus extremely badly chosen. This serves as an indication of the importance of

TABLE II  
OPTIMIZATION OF UNEQUALLY SPACED SYMMETRIC  $N$ -ELEMENT LINEAR BROADSIDE ARRAYS

(a) $N$ even										
$N$	$N_{\text{pat}}$	Optimized values						Max. residual [dB]	First null [deg]	
		$x_1$ [ $\lambda$ ]	$x_2$ [ $\lambda$ ]	$x_3$ [ $\lambda$ ]	$x_4$ [ $\lambda$ ]	$x_5$ [ $\lambda$ ]	$x_6$ [ $\lambda$ ]		Optimized array	Chebyshev array
4	12	0.19548						-15.496	~ 32.0	34.053
6	17	0.22805	0.42630					-16.914	~ 21.0	21.937
8	23	0.18964	0.45393	0.47595				-18.393	~ 16.0	16.558
10	27	0.21464	0.38517	0.46147	0.52586			-19.719	~ 13.0	13.468
12	31	0.18461	0.41678	0.41094	0.46928	0.57331		-20.911	~ 11.5	11.446
14	33	0.18837	0.39646	0.39410	0.44854	0.47684	0.61774	-21.998	~ 10.0	10.012

(b) $N$ odd										
$N$	$N_{\text{pat}}$	Optimized values						Max. residual [dB]	First null [deg]	
		$x_1$ [ $\lambda$ ]	$x_2$ [ $\lambda$ ]	$x_3$ [ $\lambda$ ]	$x_4$ [ $\lambda$ ]	$x_5$ [ $\lambda$ ]	$x_6$ [ $\lambda$ ]		Optimized array	Chebyshev array
5	10	0.47097						-13.618	~ 24.5	25.529
7	20	0.43198	0.49355					-16.321	~ 18.0	18.053
9	24	0.41587	0.43649	0.52782				-18.348	~ 14.5	14.509
11	31	0.39784	0.42441	0.44984	0.56474			-20.013	~ 12.0	12.247
13	35	0.39402	0.39114	0.44346	0.46529	0.60201		-21.324	~ 10.5	10.630
15	39	0.37362	0.41130	0.38244	0.46963	0.47979	0.63953	-22.490	~ 9.5	9.438

choosing good starting points in optimization problems where local optima may occur.

The general trend in Table I is, however, that convergence to the minimax optimum is obtained to a high degree of accuracy with a very limited number of pattern evaluations. The test examples considered may serve as a future basis of comparison with other minimax optimization algorithms.

## VI. OPTIMIZATION OF NONUNIFORMLY SPACED ARRAYS

We now consider  $N$ -element arrays with uniform excitation, i.e., all excitation coefficients  $a_i$  are chosen equal to unity. From a technical point of view, uniform excitation is much simpler to realize than the Dolph-Chebyshev excitation or any other tapered excitation. Again the element spacings as defined by (11) are the optimization parameters. The patterns are sampled in steps of  $0.5^\circ$ ,  $\theta_1$  being chosen sufficiently small to ensure that all sidelobes are included. If we take the case with  $N = 15$ ,  $\theta_1$  has been chosen to  $9^\circ$  and consequently the number of sample points  $m = 163$ .

A series of pattern synthesis examples has been carried out with  $N$  ranging from 4 to 15. For each  $N$  we start the optimization from a  $\lambda/2$ -spaced array<sup>4</sup> and continue the iterations until

$$\|h_k\| < 10^{-6} \|x_{k+1}\| \quad (16)$$

which produces sidelobes that are identical to approximately 10 significant digits.

Results are presented in Table II(a) for  $N$  even and in Table II(b) for  $N$  odd. The number of required antenna pattern evaluations  $N_{\text{pat}}$  is seen to be very moderate, one pattern evaluation corresponding to one evaluation of (12) for  $j = 1, 2, \dots, m$ , i.e.,

evaluation of (10) for  $m + 1$  values of  $\theta$ . For  $N = 15$  the optimization required a total CPU time of 5.6 s using double precision Fortran IV on an IBM 370/175 machine. Of this time, 1.8 s were spent computing the residuals (12), and 2.4 s were spent solving the linear subproblems (6). Fig. 3 shows the optimized radiation patterns as well as the initial patterns for  $N$  even, and Fig. 4 shows the corresponding patterns for  $N$  odd.

To give an impression of the overall convergence, the maximum value of the residuals (12) is plotted in Fig. 5 as a function of the number of pattern evaluations for  $N = 14$  and 15. The plateau on both curves at the beginning is a result of initially approximating the derivative matrix  $\mathcal{B}_k$  by finite differences. This requires  $n + 1$  pattern evaluations in which improvement on the maximum residual is not attempted.

Some comments on the results are in order. In Table II the position of the first null for the optimized patterns as found by inspection of Figs. 3 and 4 is compared to the position of the first null for a  $\lambda/2$ -spaced Dolph-Chebyshev array with the same number of elements, same total length, and same sidelobe level [13]. It is interesting to observe that the optimized patterns, although presumably not being expressible in terms of Chebyshev polynomials, exhibit in general main beam widths very close to those of the conventional arrays.

The initial patterns for  $N$  even have, by obvious reasons, nulls in the direction  $\theta = 90^\circ$ . In all cases considered, however, an extra sidelobe developed in that direction during the optimization, such that the total number of (equal) sidelobes became  $n + 1$ . The initial patterns for  $N$  odd have, of course, a sidelobe in the direction  $\theta = 90^\circ$ . During the optimization this particular sidelobe is shifted in position and the final pattern ends up with  $n + 1$  sidelobes. These observations agree with fundamentals from minimax approximation theory according to which solu-

<sup>4</sup> These starting points seem to be the most natural since we have no *a priori* knowledge of the optimum element distribution.

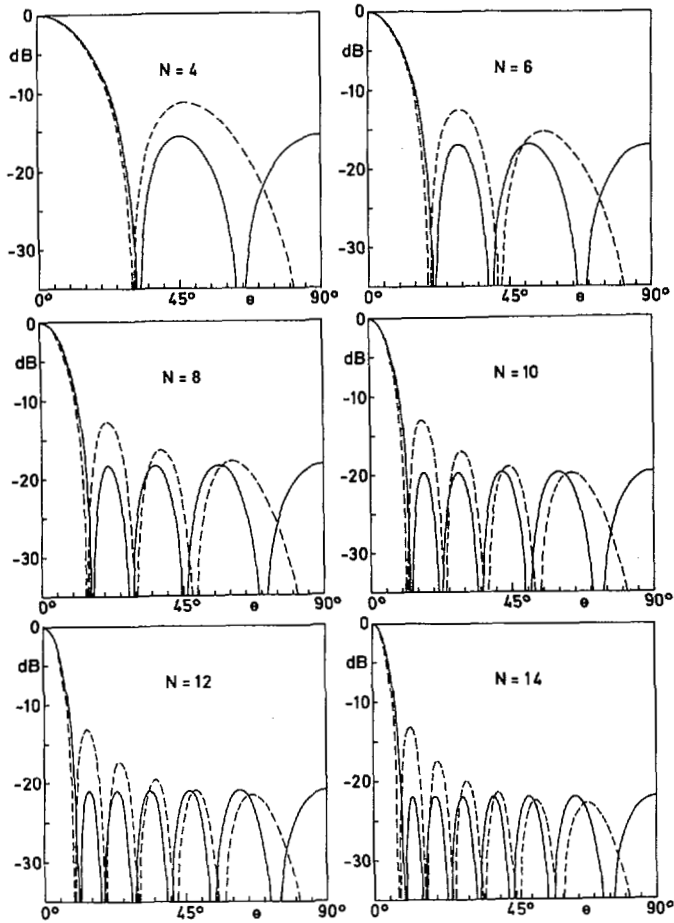


Fig. 3. Initial patterns (dashed) and optimized patterns (solid) of unequally spaced symmetric  $N$ -element linear broadside arrays,  $N$  even.

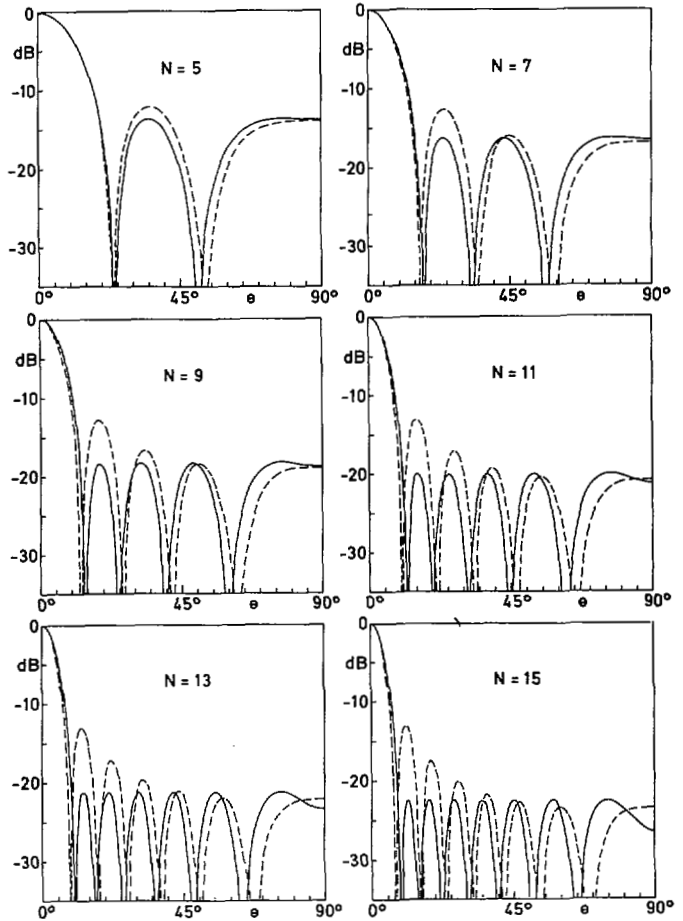


Fig. 4. Initial patterns (dashed) and optimized patterns (solid) of unequally spaced symmetric  $N$ -element linear broadside arrays,  $N$  odd.

tions to regular minimax problems with  $n$  variables attain their maximum errors in  $n + 1$  points.

One might ask the question: are the element positions presented in Table II unique or might other combinations of positions exist that would produce the same or even lower sidelobes? We do not have any final answer to this question. We have good reasons, though, to believe that the solutions are unique. For selected examples it has been tried, by choosing other initial configurations, to find other solutions. In all cases, however, the algorithm came out with the solutions in Table II.

### VII. DISCUSSION

If the element spacing in unequally spaced arrays gets too small, unwanted coupling between the elements may arise. The closest spacing in the optimized arrays presented in Table II is seen to be 0.37362 which occurs when  $N = 15$ . It is therefore relevant to try to develop minimax optimization methods which allow for constraining the optimization parameters to be larger than a certain value, or more general, to lie between preassigned limits  $a_i$  and  $b_i$

$$a_i \leq x_i \leq b_i, \quad i = 1, 2, \dots, n. \quad (17)$$

The algorithm presented in this paper seems to be particularly suitable for such an extension since constraints of the type (17) are inherently used when solving the linearized subproblems (6).

The present algorithm may also be applied to sidelobe minimization with thinned arrays. However, it is to be remembered

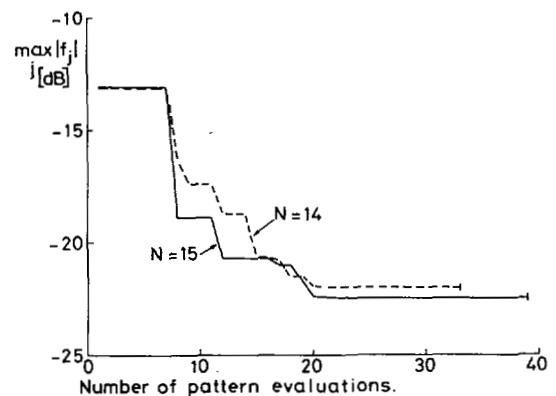


Fig. 5. Maximum residual (sidelobe level) as function of number of pattern evaluations for  $N = 14$  and  $15$ .

that with  $n$  variables only  $n + 1$  sidelobes are likely to be equalized in the minimax optimum. Since the number of sidelobes in thinned arrays is much larger than  $n + 1$ , one can easily imagine that there exist many different combinations of element spacings that produce  $n + 1$  equal sidelobes. In other words, extreme caution has to be taken against local minima. Actually we have reconsidered the 19λ aperture array with 9 elements (i.e., three variable element spacings) which Lo and Lee [14] optimized by "total enumeration." By sampling the pattern in

145 equidistantly spaced sample points in the interval  $0.052632 \leq u \leq 1.94737$ , and starting out from their best solution  $x = (0.5, 2.0, 1.5)$ , virtually no improvement was obtained. This was to be expected since the initial pattern contains already about 10 almost equal sidelobes [14, Fig. 3]. Choosing other starting points our algorithm converged to a number of local minima.<sup>3</sup>

### VIII. CONCLUSION

The general minimax antenna pattern synthesis problem has been formulated and a new nonlinear optimization algorithm with super-linear final convergence is proposed for its solution. Not requiring derivative evaluation, the proposed method is easily applied to synthesis problems with complicated pattern calculations. In this paper, Dolph-Chebyshev arrays were considered for testing purposes and also applications to sidelobe minimization in unequally spaced linear arrays with up to 15 elements by spacing variation demonstrate that the approach is very efficient and reliable. The optimized patterns are compared to conventional Chebyshev patterns and are shown to be comparable in terms of main beamwidth for the same sidelobe level. Local minima in conjunction with sidelobe minimization have been discussed.

It is believed that the present method will find wide-spread future applications in solving pattern synthesis problems that hitherto have been approached by cut-and-try methods.

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### REFERENCES

- [1] C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level," *Proc. IRE*, vol. 34, pp. 335-348, June 1946.
- [2] M. T. Ma, *Theory and application of antenna arrays*. New York: J. Wiley & Sons, Inc., 1974.
- [3] P. W. James, "Optimization of planar aerial arrays for lowest sidelobe level," *Proc. IEE*, vol. 120, pp. 1193-1199, October 1973.
- [4] R. Streit, "Sufficient conditions for the existence of optimum beam patterns for unequally spaced linear arrays with an example," *IEEE Trans. Antennas Propagat.* (Commun.), vol. AP-23, pp. 112-115, January 1975.
- [5] C. G. Broyden, "A class of methods for solving non-linear simultaneous equations," *Math. Comp.*, vol. 19, pp. 577-593, 1965.
- [6] M. J. D. Powell, "The minimax solution of linear equations subject to bounds on the variables," report C.S.S. 11, AERE, Harwell, Oxfordshire, England, December 1974.
- [7] —, "A Fortran subroutine for unconstrained minimization, requiring first derivatives of the objective functions," report R 6469, AERE, Harwell, Oxfordshire, England, July 1970.
- [8] K. Madsen, O. Nielsen, H. Schjær-Jacobsen, and L. Thrane, "Efficient minimax design of networks without using derivatives," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 803-809, October 1975.
- [9] K. Madsen, "Minimax solution of non-linear equations without calculating derivatives," to appear in *Mathematical Programming, Studies No. 3*. Amsterdam: North Holland Publishing Co., edited by M. Balinski and P. Wolfe.
- [10] K. Madsen, H. Schjær-Jacobsen, and J. Voldby, "Automated minimax design of networks," *IEEE Trans. Circuits Systems*, vol. CAS-22, pp. 791-796, October 1975.
- [11] K. Madsen and H. Schjær-Jacobsen, "Fortran subroutines for non-linear minimax optimization," report R 135, Electromagnetics Institute, Technical University of Denmark, Lyngby, Denmark, February 1975.
- [12] R. J. Stegen, "Excitation coefficients and beamwidths of Tschebyscheff arrays," *Proc. IRE*, vol. 41, pp. 1671-1674, November 1953.
- [13] H. Bach and J. E. Hansen, "Uniformly spaced arrays," Chap. 5 in *Antenna Theory*, edited by R. E. Collin and F. J. Zucker. New York: McGraw-Hill, Inc., 1969.
- [14] Y. T. Lo and S. W. Lee, "A study of space-tapered arrays," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 22-30, January 1966.

## Horizontally Polarized Waves in Inhomogeneous Media—Energy Conservation and Reciprocity Relationships

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**Abstract**—Full wave solutions for the electromagnetic fields of a horizontally polarized wave propagating through an inhomogeneous ionized medium are derived using a generalized WKB method. Both the electron density and the collision frequency of the horizontally stratified media are assumed to vary and special attention is given to permittivity profiles with critical coupling regions. The reflection and transmission coefficients and the characteristic surface impedance for an inhomogeneous layer of finite thickness are computed as functions of the transverse wave number for various permittivity profiles. Excitation of both propagating and evanescent waves are considered. For some special permittivity profiles considered, closed form analytical solutions for the electromagnetic fields are known. Computations derived from these solutions are in good agreement with those obtained using the generalized WKB method. The results are also shown to satisfy energy conservation and reciprocity relationships in electromagnetic theory.

### I. INTRODUCTION

Considerable effort has been made to derive full wave solutions for the electromagnetic fields in horizontally stratified media in terms of mathematical functions tabulated in handbooks [1]-[6]. While they do not provide solutions to the general problem in which the complex permittivity of the media is assumed to vary arbitrarily, they provide an important basis for the analysis of electromagnetic waves in inhomogeneous media. Thus using the Green's function technique, for instance, the solutions are formulated in terms of integral equations involving comparison functions that are chosen from the list of permittivity profiles for which closed form analytical solutions are known [7].

The method employed in this paper to compute horizontally polarized electromagnetic fields in an ionized media with varying electron density and collision frequency profiles is based on the conversion of Maxwell's equations into a set of loosely coupled first-order differential equations for the wave amplitudes [8]. Special attention is given to permittivity profiles with critical coupling regions where the familiar WKB approach fails.

The reflection and transmission coefficients and the characteristic surface impedance for a horizontally stratified layer of finite thickness is computed as a function of the transverse wave number and excitations of both propagating and evanescent waves are considered.

The reciprocity and realizability relationships are formulated and the numerical results are in agreement up to at least four significant figures. The computed values for the reflection and transmission coefficients and the characteristic surface impedance are also shown to be in good agreement with those derived from closed form analytical solutions for special permittivity profiles.

### II. FORMULATION OF THE PROBLEM

The electromagnetic fields due to an electric line source  $\vec{J}(\vec{r}) = J_y(x, z)\vec{a}_y$ , parallel to a horizontally stratified dielectric  $\epsilon(z)$  can be expressed in terms of a complete spectrum of horizontally polarized waves (see Fig. 1). Employing the Fourier

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