DTU Library

Gabor frames with reduced redundancy

Christensen, Ole; Rae Young, Kim; Kim, Hong Oh

Published in:
Proceeding of SAMPTA conference (published on CDROM)

Publication date:
2009

Document Version
Early version, also known as pre-print

Link back to DTU Orbit

Citation (APA):
Christensen, O., Rae Young, K., & Kim, H. O. (2009). Gabor frames with reduced redundancy. In *Proceeding of SAMPTA conference (published on CDROM)*

Gabor frames with reduced redundancy

Ole Christensen (1), Hong Oh Kim (2) and Rae Young Kim (3)

(1) Department of Mathematics, Technical University of Denmark, Building 303, 2800 Lyngby, Denmark.
(2) Department of Mathematical Sciences, KAIST, Daejeon, Korea.
(3) Department of Mathematics, Yeungnam University, Gyeongsan-si,Korea.
Ole.Christensen@mat.dtu.dk, kimhong@kaist.edu, rykim@ynu.ac.kr
This work was supported by the Korea Science and Engineering Foundation (KOSEF) Grant funded by the Korea Government(MOST)(R01-2006-000-10424-0) and by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2006-331-C00014).

Abstract:

Considering previous constructions of pairs of dual Gabor frames, we discuss ways to reduce the redundancy. The focus is on B-spline type windows.

1. Introduction

We will consider Gabor systems in $L^2(\mathbb{R})$, i.e., families of functions $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$, where

$$E_{mb}T_ng(x) := e^{2\pi imbx}g(x-na).$$

If there exists a constant $B>0$ such that

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_ng\rangle|^2 \leq B\,||f||^2,\ \forall f\in L^2(\mathbb{R}),$$

then $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ is called a Bessel sequence. If there exist two constants $A,B>0$ such that

$$A\,||f||^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_ng\rangle|^2 \leq B\,||f||^2,\ \forall f\in L^2(\mathbb{R}),$$

then $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ is called a frame. If $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ is a frame with dual frame $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$, then

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_nh\rangle E_{mb}T_ng,\ f\in L^2(\mathbb{R}),$$

where the series expansion converges unconditionally in $L^2(\mathbb{R})$.
Our starting point is the duality condition for Gabor frames, originally due to Ron and Shen [4]. We use the version due to Janssen [3]:

Lemma 1..1 *Two Bessel sequences $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ form dual Gabor frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b+k)}h(x+k) \quad = \quad b\delta_{n,0} \qquad (1..1)$$

for a.e. $x\in[0,1]$.

The Bessel condition in Lemma 1..1 is always satisfied for bounded windows with compact support, see [1]. Note that if g and h have compact support, we only need to check a finite number of conditions in (1..1). In this paper we will usually choose b so small that only the condition for $n=0$ has to be verified.

2. The range $\frac{1}{2N-1} < b < \frac{1}{N}$

We first cite a result from [2]. It yields an explicit construction of dual Gabor frames:

Theorem 2..1 *Let $N\in\mathbb{N}$. Let $g\in L^2(\mathbb{R})$ be a real-valued bounded function with supp $g\subset[0,N]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1. \qquad (2..1)$$

Let $b \in]0,\frac{1}{2N-1}]$. Consider any scalar sequence $\{a_n\}_{n=-N+1}^{N-1}$ for which

$$a_0 = b \ \text{ and } \ a_n + a_{-n} = 2b,\ n=1,2,\cdots N-1, \quad (2..2)$$

and define $h\in L^2(\mathbb{R})$ by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n). \qquad (2..3)$$

Then g and h generate dual frames $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

The above result can be extended:

Corollary 2..2 *Consider any $b\leq 1/N$. With g and a_n as in Theorem 2..1, the function*

$$h(x) = \left(\sum_{n=-N+1}^{N-1} a_n g(x+n)\right)\chi_{[0,N]}(x) \qquad (2..4)$$

is a dual frame generator of g.

Proof. Consider the condition (1..1) for $n=0$; only the values of $h(x)$ for $x\in[0,N]$ play a role, so since the condition holds for the function in (2..3), it also holds for the function in (2..4). □

The cut-off in (2..4) yields a non-smooth function. However, for any $b<1/N$, we might modify h slightly and obtain a smooth dual generator:
In particular, we obtain the following:

Corollary 2..3 *Consider any $b<1/N$, and take $\epsilon < 1/b-N$. With g as in Theorem 2..1, the function $h(x) = b, x\in[0,N]$ has an extension to a function of desired smoothness, supported on $[-\epsilon, N+\epsilon]$, which is a dual frame generator of g.*

Proof. The choice $a_n = b,\ n = -N+1,\dots,N-1$, leads to

$$\sum_{n=-N+1}^{N-1} a_n g(x+n) = b,\ x \in [0, N].$$

Given $\epsilon < 1/b - N$ and any functions $\phi_1 : [-\epsilon, 0[\to \mathbb{R}$ and $\phi_2 :]N, N+\epsilon] \to \mathbb{R}$, the function

$$h(x) = \begin{cases} \phi_1(x), & x \in [-\epsilon, 0[, \\ \sum_{n=-N+1}^{N-1} a_n g(x+n) = b, & x \in [0, N], \\ \phi_2, & x \in]N, N+\epsilon], \\ 0, & x \notin [-\epsilon, N+\epsilon], \end{cases}$$

will satisfy (1..1); in fact, for $n \neq 0$, the support of the functions $g(\cdot \pm n/b)$ and h are disjoint, and for $n = 0$ we are (for all relevant values of x) back at the function in (2..4). The functions ϕ_1 and ϕ_2 can be chosen such that the function h has the desired smoothness. □

The assumptions in Theorem 2..1 are tailored to B-splines, defined inductively by

$$B_1 := \chi_{[0,1]},\quad B_{N+1} := B_N * B_1.$$

Direct calculations shows that

$$B_2(x) = \begin{cases} x & \text{if } x \in [0,1], \\ 2-x & \text{if } x \in [1,2], \\ 0 & \text{otherwise}, \end{cases}$$

and

$$B_3(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \in [0,1], \\ -x^2+3x-\frac{3}{2} & \text{if } x \in [1,2], \\ \frac{1}{2}x^2-3x+\frac{9}{2} & \text{if } x \in [2,3], \\ 0 & \text{otherwise}. \end{cases}$$

In general, the functions B_N are $(N-2)-$times differentiable piecewise polynomials (explicit expressions are known). Furthermore, supp $B_N = [0, N]$, and the partition of unity condition (2..1) is satisfied.
In case $g = B_N$, the dual generators in Theorem 2..1 are splines, of the same smoothness as B_N itself. By compressing the function $\sum_{n=-N+1}^{N-1} a_n g(x+n)$ from the interval $[-N+1, 0]$ to $[-\epsilon, 0]$ and from $[N, 2N-1]$ to $[N, N+\epsilon]$ we obtain a dual in (2..3) with the same features:

Example 2..4 For the B-spline $B_3(x)$ and $b = 1/5$, Theorem 2..1 yields the symmetric dual

$$h_3(x) = \frac{1}{5}\begin{cases} 1/2\,x^2 + 2\,x + 2, & x \in [-2,-1[, \\ -1/2\,x^2 + 1, & x \in [-1, 0[, \\ 1, & x \in [0,3[, \\ -1/2\,x^2 + 3\,x - 7/2, & x \in [3,4[, \\ 1/2\,x^2 - 5x + 25/2, & x \in [4,5[, \\ 0, & x \notin [0,5[. \end{cases} \tag{2..5}$$

See Figure 1.
Now, for $b = 1/4$, we can use Corollary 2..3 for $\epsilon < 4 - 3 = 1$. Taking $\epsilon = 1/2$, we compress the function

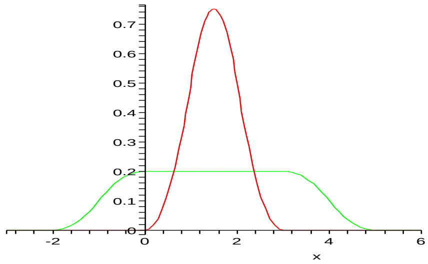

Figure 1: B_3 and the dual generator h_3 in (2..5).

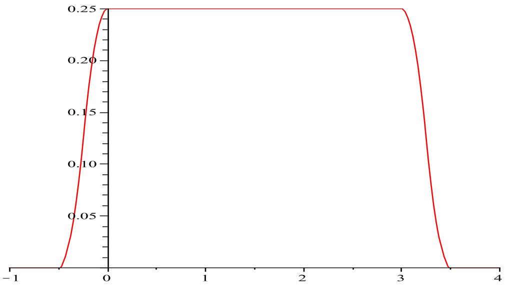

Figure 2: The function h in (3..13)..

h_3 in (2..5) from $[-2, 0]$ to $[-1/2, 0]$ and from $[3, 5]$ to $[3, 31/2]$ and obtain the dual

$$h(x) = \frac{1}{4}\begin{cases} 1/2\,(4x)^2 + 2\,(4x) + 2, & x \in [-1/2, -1/4[, \\ -1/2\,(4x)^2 + 1, & x \in [-1/4, 0[, \\ 1, & x \in [0,3[, \\ -1/2\,(4(x-3)+3)^2 + 3\,(4(x-3)+3) - 7/2, & x \in [3, 3+1/4[, \\ 1/2\,(4(x-3)+3)^2 - 5(4(x-3)+3) + 25/2, & x \in [3+1/4, 3+1/2[, \\ 0, & x \notin [-1/2, 3+1/2[. \end{cases}$$

$$= \frac{1}{4}\begin{cases} 8\,x^2 + 8\,x + 2, & x \in [-1/2, -1/4[, \\ -8\,x^2 + 1, & x \in [-1/4, 0[, \\ 1, & x \in [0,3[, \\ -8\,x^2 + 48\,x - 71, & x \in [3, 3+1/4[, \\ 8\,x^2 - 56\,x + 98, & x \in [3+1/4, 3+1/2[, \\ 0, & x \notin [-1/2, 3+1/2[. \end{cases}$$

See Figure 2. □

3. B_2 and $1/2 < b < 1$

In the following discussion, we consider dual windows associated with a Gabor frame $\{E_{mb}T_nB_2\}_{m,n\in\mathbb{Z}}$ generated by the B-spline B_2. The arguments can be extended to general functions supported on $[0, 2]$. Take any function h with values specified only on $[0, 2]$ and such that

$$\sum_{k\in\mathbb{Z}} B_2(x+k)h(x+k) = 1,\ x \in [0, 1]. \tag{3..1}$$

In fact, due to the support of B_2, only the values for $h(x)$ for $x \in [0, 2]$ play a role for that condition. We know that

for any $b \le 1/2$ the function generates – up to a certain scalar multiple – a dual of g.
Now consider any $1/2 < b < 1$; that is, we have $1 < 1/b < 2$.

Lemma 3..1 *Assume that $h(x),\ x \in [0,2]$ is chosen such that* (3..1) *is satisfied. The the following hold:*

(i) If

$$\sum_{k\in\mathbb{Z}} B_2(x-1/b+k)h(x+k)=0,\ x\in\mathbb{R}, \quad (3..2)$$

and

$$\sum_{k\in\mathbb{Z}} B_2(x+1/b+k)h(x+k)=0,\ x\in\mathbb{R}, \quad (3..3)$$

then

$$B_2(x-1/b)h(x)+B_2(x-1/b+1)h(x+1)=0,$$

$$x\in[1/b,2], \quad (3..4)$$

$$B_2(x+1/b-1)h(x-1)+B_2(x+1/b)h(x)=0$$

$$x\in[0,2-1/b]. \quad (3..5)$$

These equations determine $h(x)$ for

$$x\in[-1,1-1/b]\cup[1+1/b,3].$$

(ii) If $h(x)$ for $x\in[-1,1-1/b]\cup[1+1/b,3]$ is chosen such that (3..4) *and* (3..5) *are satisfied, and*

$$h(x)=0, x\notin[0,2]\cup[-1,1-1/b]\cup[1+1/b,3],$$

then (3..2) *and* (3..3) *hold.*

Proof. We consider (3..2) for $x\in[1,2]$, and split into two cases:
For $x\in[1,1/b]$, (3..2) yields that

$$0=B_2(x-1/b+1)h(x+1)+B_2(x-1/b+2)h(x+2); \quad (3..6)$$

the equation only involve $h(x)$ for

$$x\in[2,1+1/b]\cup[3,2+1/b].$$

For $x\in[1/b,2]$, (3..2) yields that

$$0=B_2(x-1/b)h(x)+B_2(x-1/b+1)h(x+1);$$

since $h(x)$ is known, this implies that

$$h(x+1)=\frac{-B_2(x-1/b)h(x)}{B_2(x-1/b+1)},\ x\in[1/b,2],$$

that is,

$$h(x)=\frac{-B_2(x-1/b-1)h(x-1)}{B_2(x-1/b)},\ x\in[1/b+1,3].$$

Similarly, considering (3..3) for

$$x\in[0,1]=[0,2-1/b]\cup[2-1/b,1]$$

leads to (3..5) and

$$B_2(x+1/b-2)h(x-2)+B_2(x+1/b-1)h(x-1)$$

$$=0,\ x\in[2-1/b,1]; \quad (3..7)$$

the equation (3..7) only involves $h(x)$ for

$$x\in[-1/b,-1]\cup[1-1/b,0],$$

and (3..5) implies that

$$h(x-1)=\frac{-B_2(x+1/b)h(x)}{B_2(x+1/b-1)},\ x\in[0,2-1/b],$$

i.e.,

$$h(x)=\frac{-B_2(x+1/b+1)h(x+1)}{B_2(x+1/b)},\ x\in[-1,1-1/b].$$

For the proof of (ii), the condition

$$h(x)=0, x\notin[0,2]\cup[-1,1-1/b]\cup[1+1/b,3],$$

implies that (3..6) and (3..7) are satisfied. By construction, (3..2) and (3..3) are satisfied. □

Lemma 3..1 shows that if we want that (3..1), (3..2), and (3..3) hold for some $b\in]1/2,1]$, then h in general will take values outside $[0,2]$. However, the proof shows that we under certain circumstances can find a solution h having support in $[0,2]$. In that case, the support will actually be a subset of $[0,2]$:

Corollary 3..2 *Let $b\in]1/2,1]$. Assume that supp $h\subseteq[0,2]$ and that* (3..1) *and* (3..2) *holds. Then*

$$h(x)=0,\ x\in[0,2-1/b]\cup[1/b,2]. \quad (3..8)$$

Proof. According to the proof of Lemma 3..1, we obtain that $h(x)=0$ on $[1/b+1,3]$ by requiring that $h(x)=0$ for $x\in[1/b,2]$; and we obtain that $h(x)=0$ on $[-1,1-1/b]$ by requiring that $h(x)=0$ for $x\in[0,2-1/b]$. □

If supp $h\subseteq[0,2]$, the condition (3..8) implies that h at most can be nonzero on the interval $[2-1/b,1/b]$ having length $2/b-2$. In order for (3..1) to hold, this interval must have length at least 1; thus, we need to consider b such that $2/b-2\ge 1$, i.e., $b\le 2/3$. Note that if $b\le 2/3$, then $2/b\ge 3$: that is, because B_2 and h are supported on $[0,2]$, Janssen's duality conditions in (1..1) are automatically satisfied for $n=\pm2,\pm3,\ldots$.

Corollary 3..3 *Consider $b\in]1/2,2/3]$. Then there exists a function h with supp $h\subseteq[0,2]$ such that* (3..1) *and* (3..2) *hold; and $bh(x)$ is a dual generator of B_2 for these values of b.*

Proof. For $x \in [0, 2-1/b] \cup [1/b, 2]$, let $h(x) = 0$. For $x \in [0,1]$, the equation (3..1) means that

$$xh(x) + (1-x)h(x+1) = 1.$$

This implies that

$$\begin{aligned} xh(x) &= 1, \ x \in [1/b - 1, 1], \\ (1-x)h(x+1) &= 1, \ x \in [0, 2-1/b]; \end{aligned}$$

that is,

$$h(x) = \frac{1}{x}, \ x \in [1/b - 1, 1], \tag{3..9}$$

and

$$h(x) = \frac{1}{2-x}, \ x \in [1, 3 - 1/b]. \tag{3..10}$$

Finally, for $x \in [2 - 1/b, 1/b - 1]$ and $x \in [3 - 1/b, 1/b]$, choose $h(x)$ such that

$$xh(x) + (1-x)h(x+1) = 1.$$

By construction, $bh(x)$ is a dual generator. □

For $b = 3/5$ we will now explicitly construct a continuous dual generator h of B_2 with support in $[0,2]$. Putting Corollary 3..2, (3..9), and (3..10) together, we can state a result about how a dual window supported on $[0,2]$ must look like on parts of $[0,2]$:

Lemma 3..4 *For $b = 3/5$, every dual generator of B_2 with support in $[0,2]$ has the form*

$$h(x) = \begin{cases} 0 & \text{if } x \leq 1/3; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ 0 & \text{if } x \geq 5/3. \end{cases}$$

That is, we only have freedom on the definition of h on $]1/3, 2/3[\cup]4/3, 5/3[$.

Note that on $[2/3, 4/3]$, the function h is symmetric around $x = 1$. We will now show that it is possible to define h on $]1/3, 2/3[\cup]4/3, 5/3[$ in such a way that h becomes symmetric around $x = 1$.
First, we note that this form of symmetry means that

$$h(1-x) = h(1+x), \ x \in]1/3, 2/3[. \tag{3..11}$$

Put together with the duality condition, we thus require that

$$xh(x) = 1 - (1-x)h(1-x), \ x \in]1/3, 2/3[. \tag{3..12}$$

The condition (3..12) shows that must define $h(1/2) = 1$. Now, taking any continuous function h defined on $[1/3, 1/2]$ with the properties that $h(1/3) = 0$ and $h(1/2) = 1$, the condition (3..12) shows how to define $h(x)$ on $]1/2, 2/3[$; and, finally, the condition (3..11) shows how to define h on $]4/3, 5/3[$ such that the resulting function is a symmetric dual generator.

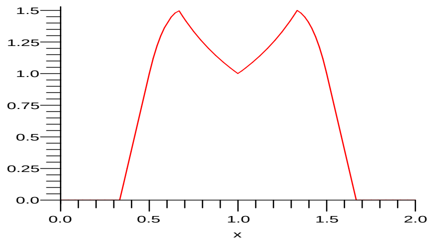

Figure 3: The function h in (3..13)..

Put

$$h(x) = 6x - 2, \ x \in [1/3, 1/2].$$

Then, for $x \in [1/2, 2/3]$,

$$\begin{aligned} h(x) &= \frac{1 - (1-x)h(1-x)}{x} \\ &= \frac{-6x^2 + 10x - 3}{x}. \end{aligned}$$

The condition $h(1+x) = h(1-x), \ x \in]1/3, 2/3[$ can also be expressed as $h(x) = h(2-x), \ x \in]4/3, 5/3[$. Thus, for $x \in [4/3, 3/2]$ we arrive at

$$h(x) = h(2-x) = \frac{-6x^2 + 14x - 7}{2-x}, \ x \in [4/3, 3/2];$$

while, for $x \in [3/2, 5/3]$,

$$h(x) = h(2-x) = 6(2-x) - 2 = 10 - 6x.$$

We have arrived at the following conclusion:

Lemma 3..5 *For $b = 3/5$, the function*

$$h(x) = \begin{cases} 0 & \text{if } x \leq 1/3; \\ 6x - 2 & \text{if } x \in [1/3, 1/2]; \\ \frac{-6x^2+10x-3}{x} & \text{if } x \in [1/2, 2/3]; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ \frac{-6x^2+14x-7}{2-x} & \text{if } x \in [4/3, 3/2]; \\ 10 - 6x & \text{if } x \in [3/2, 5/3]; \\ 0 & \text{if } x \geq 5/3 \end{cases} \tag{3..13}$$

is a continuous symmetric dual generator of B_2.

References:

[1] Christensen, O.: *Frames and bases. An introductory course.* Birkhäuser 2007.

[2] Christensen, O. and Kim, R. Y.: *On dual Gabor frame pairs generated by polynomials.* J. Fourier Anal. Appl., accepted for publication.

[3] Janssen, A.J.E.M.: *The duality condition for Weyl-Heisenberg frames.* In "Gabor analysis: theory and applications" (eds. H.G. Feichtinger and T. Strohmer). Birkhäuser, Boston, 1998.

[4] Ron, A. and Shen, Z.: *Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$.* Canad. J. Math. **47** no. 5 (1995), 1051–1094.