Computer modelling of terrestrial gamma-radiation fields

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Computer Modelling of Terrestrial Gamma-Radiation Fields

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Computer Modelling of Terrestrial Gamma-Radiation Fields

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Abstract

The plane, one-dimensional photon transport equation is considered in the case of two adjacent media, where one medium contains the γ-ray sources and the other is inactive. Assuming that the sources have a composite line spectrum, and that they are uniformly distributed, an explicit solution is given for the uncollided flux component. The scattered component is evaluated on the basis of the double-P1 approximation, involving separate treatments of the upstreaming and the downstreaming flux, and a numerical method for solution of the corresponding equations is presented. The computational method permits determination of the differential and angular energy flux throughout the inactive medium. Formulas for obtaining integral field quantities (scalar energy flux, scalar number flux, and absorbed dose rate) are given. The flux calculation method is used in conjunction with a data processing system for evaluation of terrestrial gamma-radiation fields. A detailed description is given of both the data files and the programs of which the system consists. To illustrate the performance of the system, results obtained for the radiation field in water above sand are presented.
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1. INTRODUCTION

In this report we describe techniques for computational evaluation of terrestrial gamma-radiation fields, i.e. environmental radiation fields produced by the natural gamma-ray emitters (K and members of the thorium and the uranium decay series) in the crustal materials of the earth. The techniques are currently being used as an aid to the interpretation of radiometric surveys of geologic formations. The methods developed, and the results selected to illustrate their performance, are considered to be a valuable supplement to the similar, though more comprehensive achievements of Beck and co-workers at the USAEC Health and Safety Laboratory. Thus we have independently solved a one-dimensional, two-media photon transport problem, and the solution has been used for compilation of data on terrestrial gamma-radiation fields in water.

The basic idealization made in this work is indicated in fig. 1. Two semi-infinite, homogeneous media, I and II, border on each other along a plane interface a-a. Medium I contains spatially uniform gamma-ray emitters having a composite line spectrum in the general case. We shall focus our interest on prediction of the photon flux at selected heights in medium II (z > 0), although it will be necessary to work out solutions for the radiation field in both media as a whole.

In a previous work the same problem was solved by means of the double-P polynomial expansion method for the more simple case in which medium II is a vacuum. The double-P approximation proved to yield an efficient and reasonably accurate flux calculation for semi-infinite, plane-geometry conditions, and the present work is therefore based on a further development of this technique. In the same way as in the formulation and solution of the double-P equations was influenced by Gerstl, who considered a finite slab with a source at one end-face.

In the following we shall first (chapter 2) set up the plane, one-dimensional transport equation satisfied by the total flux of photons, uncollied as well as scattered, and give expressions for the uncollied flux. Chapter 3 is devoted to the double-P approximation, where the components of the transport equation for the scattered flux are expanded in half-range Legendre polynomials through the first-order terms; an equation system for the expansion coefficients is established in section 3.1, and in section 3.2 its numerical solution is discussed. Formulas for integral field quantities (e.g. scalar number flux and absorbed dose rate) derived from the calculated angular flux densities are given in chapter 4.
Two computer programs were written to carry out the flux calculations delineated in chapters 2-4. They are part of a data processing system, which in addition covers three editing programs, three files with basic data and one with angular flux results. By means of this system, which is described in chapter 5, radiation field calculations can be accomplished for any combination of medium I and medium II.

Chapter 6 can be read independently. Tables and graphs are presented illustrating the flux distributions in water resulting from the natural radioactivity of the underlying material.

2. THE TRANSPORT EQUATION

Several computational advantages are obtained by formulating the plane one-dimensional photon transport equation in terms of wavelength and energy flux in preference to energy and number flux:

\[
\frac{\partial}{\partial z} \int_{\lambda_2}^{\lambda_1} I(z, \omega, \lambda) + \mu(z, \lambda) \frac{1}{4\pi} \int d\Omega' d\lambda' \frac{\delta(1 + \chi' - \frac{\lambda}{\lambda'})}{2\pi} \delta(1 + \chi' - \frac{\lambda}{\lambda'}) + \frac{E q(z, \lambda)}{4\pi} (-\infty < z < \infty),
\]

where

\( I(z, \omega, \lambda) \) = angular energy flux of photons (Mev cm\(^{-2}\) s\(^{-1}\) Mev\(^{-1}\) stered\(^{-1}\))
\( z \) = distance along the z-axis, cf. fig. 1 (cm)
\( \omega \) = unit vector in the direction of photon movement
\( \lambda \) = wavelength of the radiation in units of Compton wavelength
\( \mu(z, \lambda) \) = total macroscopic cross section without coherent scattering (cm\(^{-1}\))
\( E \) = energy (Mev), and
\( q(z, \lambda) \) = position and wavelength distribution of isotropically radiating source (photons cm\(^{-3}\) Mev\(^{-1}\)).

Dashed symbols refer to conditions prior to photon scattering. The Klein-Nishina scattering kernel \( k(\chi', \chi) \) has the expression

\[
k(\chi', \chi) = \begin{cases} n_e(z) \sigma_T \frac{3\chi'}{8} \left[ \left( \frac{\chi'}{\chi} + 1 \right) + 2(1 - \chi') + (\chi - \chi')^2 \right], & \chi - 2 \leq \chi' \leq \chi; \\ 0, & \text{otherwise}; \end{cases}
\]

where \( n_e(z) \) is the electron density (cm\(^{-3}\)), and \( \sigma_T = \frac{8\pi\sigma_0^2}{3} \) is the Thomson cross section; inserting the value \( r_o = 0.281776 \times 10^{-12} \) cm for the classical electron radius, we find \( \sigma_0 = 0.66516 \) barns/electron.

Specializing to the present two-media problem (fig. 1), the source term may be written

\[
q(z, \lambda) = H(-z) q(\lambda),
\]

where \( H \) is Heaviside’s step function. Further, the cross section \( \mu(z, \lambda) \) becomes a piecewise constant function of \( z \):

\[
\mu(z, \lambda) = \begin{cases} \mu_I(\lambda), & z < 0 \\ \mu_M(\lambda), & z > 0 \end{cases}
\]

The energy flux \( I \) may be split into an uncollided part \( U \) and a scattered part \( \psi \):

\[
I(z, \omega, \lambda) = U(z, \omega, \lambda) + \psi(z, \omega, \lambda).
\]

The uncollided component \( U \) satisfies the scattering-free transport equation

\[
\omega \frac{\partial}{\partial z} U(z, \omega, \lambda) + \mu(z, \lambda) U(z, \omega, \lambda) = \frac{E H(-z) q(\lambda)}{4\pi} (-\infty < z < \infty),
\]

whereas the equation for the scattered component \( \psi \) is characterized by a source term equal to the density of first-collisions:
\[
\omega \frac{\partial}{\partial z} \psi(z, \omega, \lambda) + \mu(z, \lambda) \psi(z, \omega, \lambda) =
\]
\[\int \int \left[ \psi(z, \omega', \lambda') + \Omega(z, \omega', \lambda') \right] k(\lambda, \lambda') \frac{\Gamma(\lambda' - \lambda - \Omega')}{2\pi} d\Omega' d\lambda'.
\]
\[(-\infty < z < \infty).\]

The proper boundary conditions are similar for eqs. (6) and (7):

(a) the flux at the boundary must be continuous with respect to \(z\):
\[
U(-0, \omega, \lambda) = U(+0, \omega, \lambda)
\]
and
\[
\psi(-0, \omega, \lambda) = \psi(+0, \omega, \lambda)
\]

(b) the flux must be finite in both limits:
\[
U(\pm \infty, \omega, \lambda) < \infty
\]
\[
\psi(\pm \infty, \omega, \lambda) < \infty.
\]

The solution for \(U\) is easily constructed and is given in the following scheme:

\[
U(z, \omega, \lambda) =
\]

<table>
<thead>
<tr>
<th>(z) &lt; 0</th>
<th>(\omega) &gt; 0 (upgoing)</th>
<th>(\omega) &lt; 0 (downgoing)</th>
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<tr>
<td>(I)</td>
<td>(p(\lambda))</td>
<td>(p(\lambda)[1-\exp(-\frac{\mu(\lambda)}{\omega} z)])</td>
</tr>
<tr>
<td>(II)</td>
<td>(p(\lambda) \exp(-\frac{\mu(\lambda)}{\omega} z))</td>
<td>0</td>
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with
\[
p(\lambda) = \frac{E \varphi(\lambda)}{4\pi \mu_0(\lambda)}.
\]

An approximate solution for \(\psi\) using double-\(P_1\) expansions is developed in the following chapter.

---

### 3. DOUBLE-\(P_1\) APPROXIMATION

The basic idea in double-\(P_1\) technique is to expand the up-streaming flux \((\omega) > 0\) and the down-streaming flux \((\omega) < 0\) into separate half-range spherical harmonics. Such a procedure seems natural to apply to the present problem in view of the different expressions for the direct flux \(U\) when \((\omega) > 0\) and when \((\omega) < 0\) (eq. (10)).

#### 3.1. Derivation of Equations

We define
\[
\psi^\pm(z, \omega, \lambda) = \psi(z, \omega, \lambda) H(\pm \omega)
\]
\[U^\pm(z, \omega, \lambda) = U(z, \omega, \lambda) H(\pm \omega).
\]

For any \(\omega\) we have \(\phi = \phi^+ + \phi^-\) and \(U = U^+ + U^-\); the expressions for \(U^+\) and \(U^-\) follow from (10). Eq. (7) can be written
\[
\omega \frac{\partial \psi^\pm(z, \omega, \lambda)}{\partial z} + \mu(z, \lambda) \psi^\pm(z, \omega, \lambda) =
\]
\[\int \int \left( \psi^+ U^+ + \psi^- U^- \right)(z, \omega', \lambda') k(\lambda, \lambda') \frac{\Gamma(\lambda' - \lambda - \Omega')}{2\pi} d\Omega' d\lambda',
\]
where the upper sign is taken for \((\omega) > 0\), the lower for \((\omega) < 0\). The boundary conditions are
\[
\psi^\pm(-0, \omega, \lambda) = \psi^\pm(+0, \omega, \lambda),
\]
and
\[
\psi^\pm(\pm \infty, \omega, \lambda) < \infty.
\]

For \(0 \leq \omega \leq 1\) the general half-range spherical harmonics expansion is
\[
\varphi(\omega) = \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell P_\ell(2\omega-1), \quad a_\ell = \int_0^1 \varphi(\omega) P_\ell(2\omega-1) d\omega.
\]
and for \(-1 < \omega < 0\)

\[
\phi(\omega) = \sum_{\ell=0}^{\infty} (2\ell+1) a_{\ell} P_{\ell}(2\omega+1),
\]

where \(P_{\ell}\) stands for the usual Legendre polynomial of order \(\ell\). Making the abbreviations

\[
P_{\ell}^\pm(\omega) \equiv P_{\ell}(2\omega+1) H(\pm \omega),
\]

we expand the different components of eq. (13) according to (16) and (17):

\[
\psi_{\ell}^\pm(z, \omega, \lambda) = \sum_{\ell=0}^{\infty} (2\ell+1) \psi_{\ell}^\pm(z, \lambda) P_{\ell}^\pm(\omega),
\]

with

\[
\psi_{\ell}^\pm(z, \lambda) = \int_{-1}^{1} \psi(z, \omega, \lambda) P_{\ell}(\omega) d\omega,
\]

and

\[
U_{\ell}^\pm(z, \omega, \lambda) = \sum_{\ell=0}^{\infty} (2\ell+1) U_{\ell}^\pm(z, \lambda) P_{\ell}^\pm(\omega),
\]

with

\[
U_{\ell}^\pm(z, \lambda) = \int_{-1}^{1} U(z, \omega, \lambda) P_{\ell}(\omega) d\omega.
\]

Orthogonality and recursion relations for the half-range spherical harmonics \(P_{\ell}^\pm(\omega)\) are given in Appendix I. If the expansions above are substituted in (13), and Compton's angular scattering kernel

\[
b(1 + \lambda' - \lambda, -\omega, \Omega')/2\pi \text{ is expanded in full-range spherical harmonics in } \omega \cdot \Omega',\]

it is possible after some reductions \(^6,7\) to obtain an infinite set of interlinked integro-differential equations satisfied by the expansion coefficients \(\psi_{\ell}^\pm(z, \lambda)\) for the scattered flux \(\psi\):

\[
\frac{\partial \psi_{\ell}^\pm(z, \lambda)}{\partial z} + \frac{\partial \psi_{\ell}^\pm(z, \lambda)}{\partial \omega} + \frac{\partial \psi_{\ell}^\pm(z, \lambda)}{\partial \lambda} = \sum_{\ell' = 0}^{\infty} 2(2\ell'+1) \frac{c_{\ell\ell'}}{2\pi} \int k(\lambda', \lambda) P_{\lambda'}(1+\lambda'-\lambda) \sum_{\ell''=0}^{\infty} (2\ell''+1) \right\{ (\psi_{\ell''}^+, U_{\ell''}^+)(z,\lambda') c_{\ell''}\lambda'}

\[+ (\psi_{\ell''}^-, U_{\ell''}^-)(z,\lambda') c_{\ell''}\lambda'} \right\} d\lambda'.
\]

From this point it becomes necessary to distinguish between the cases \(z < 0\) and \(z > 0\). We put \(\mu_1 = \mu_1(\lambda')\), \(\mu_2 = \mu_2(\lambda')\), introduce the auxiliary quantities \(V_{\ell}(\lambda) = \int_{-1}^{1} P_{\ell}^+(\omega) \exp(-\omega/\omega) d\omega\) (see Appendix III) and find

1) \(z < 0\):

\[
U_{\ell}^- = p(\lambda') \left( \int \phi_{\ell}^-(\omega) \left[ 1 - e^{-\mu_1 z/\omega} \right] d\omega \right) = p(\lambda') \left[ \delta_{\ell0} - (-1)^\ell V_{\ell}(\mu_1 z/\omega) \right],
\]

and

\[
U_{\ell}^+ = p(\lambda') \left( \int \phi_{\ell}^+(\omega) d\omega \right) = p(\lambda') \delta_{\ell0}.
\]

2) \(z > 0\):

\[
U_{\ell}^- = p(\lambda') \left( \int \phi_{\ell}^-(\omega) \exp(-\omega z/\omega) d\omega \right) = p(\lambda') V_{\ell}(\mu_1 z/\omega) \delta_{\ell0},
\]

and

\[
U_{\ell}^+ = 0.
\]
the right side of (20) =
\[
\sum_{n=0}^{\infty} \frac{2n+1}{2} c_{nt} \int_{-2}^{\lambda} k(\lambda', \lambda) \, P_n(t + \lambda' - \lambda) \, d\lambda'.
\]
(26)

Henceforward we shall be concerned with the consistent double-P\(_1\) approximation, i.e. we postulate \(\mathcal{V} < 1\). Then the equation valid for \(z < 0\) ((20), (23)) and that valid for \(z > 0\) ((20), (26)) each reduces to a set of four coupled differential equations:

\[
D^0_0(z, \lambda) + \mu_0(z) \psi^0_0(z, \lambda) = \int_{-2}^{\lambda} k(\lambda', \lambda) \, p(\lambda') \, d\lambda' + \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{nt} \int_{-2}^{\lambda} P_n(1 + \lambda' - \lambda) \, k(\lambda', \lambda) \sum_{\ell=0}^{\infty} (2\ell+1) \left[ c^*_{nt} \psi^*_\ell(z, \lambda') + \psi_\ell(z, \lambda') c^*_{nt} \right] \, d\lambda'.
\]
(27)

\[
c_{nt} \left( \psi_\ell(z, \lambda') - (-1)^{\ell} p(\lambda') \, V_\ell(\mu^2 \lambda') \right) \, d\lambda',
\]

\[
D^\pm_1(z, \lambda) + \mu_1(z) \psi^\pm_1(z, \lambda) = \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{nt} \int_{-2}^{\lambda} P_n(1 + \lambda' - \lambda) \, k(\lambda', \lambda) \sum_{\ell=0}^{\infty} (2\ell+1) \left[ c^*_{nt} \psi^*_\ell(z, \lambda') + \psi_\ell(z, \lambda') c^*_{nt} \right] \, d\lambda'.
\]
(28)

\[
c_{nt} \left( \psi_\ell(z, \lambda') - (-1)^{\ell} p(\lambda') \, V_\ell(\mu^2 \lambda') \right) \, d\lambda',
\]

both valid for \(z < 0\), and

\[
D^\pm_0(z, \lambda) + \mu_0(z) \psi^\pm_0(z, \lambda) = \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{nt} \int_{-2}^{\lambda} P_n(1 + \lambda' - \lambda) \, k(\lambda', \lambda) \sum_{\ell=0}^{\infty} (2\ell+1) \left[ c^*_{nt} \psi^*_\ell(z, \lambda') + \psi_\ell(z, \lambda') c^*_{nt} \right] \, d\lambda'.
\]
(29)

\[
p(\lambda') \, V_\ell(\mu^2 \lambda') + c_{nt} \left( \psi_\ell(z, \lambda') \right) \, d\lambda',
\]

both valid for \(z > 0\). In these expressions we have used the abbreviations

\[
\psi_\ell(z, \lambda') = \int_{-2}^{\lambda} k(\lambda', \lambda) \, p(\lambda') \, d\lambda'.
\]

We shall now assume that the source is a spectrum containing \(P\) lines:

\[
q(\lambda) = \sum_{p=1}^{P} Q_p \, \delta(\lambda - \lambda_p).
\]
(33)

\(Q_p\) is the intensity (photons cm\(^{-3}\) s\(^{-1}\)) of line no. \(p\) with the energy \(E_p\) and the wavelength \(\lambda_p\) = \(f_p / E_p\) (\(f_p = 0.5110058\)). According to (11), for an arbitrary function \(G(\lambda', \lambda)\) we have

\[
\int_{-2}^{\lambda} G(\lambda', \lambda) \, p(\lambda') \, d\lambda' = \sum_{p=1}^{P} \frac{Q_p}{4\pi \mu^2 \lambda_p} G(\lambda''(p), \lambda),
\]
(34)

where \(\mu^2(p) = \mu^2(\lambda_p))\). The summation is extended over all lines in the integration interval from \(\lambda - 2\) to \(\lambda\).

The equations (27) - (30) can be summarized as
\[
\begin{bmatrix}
\frac{1}{2} & 1 & 0 & 0 \\
1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 1 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\phi_0^+(z, \lambda) \\
\phi_1^+(z, \lambda) \\
\phi_0^-(z, \lambda) \\
\phi_1^-(z, \lambda)
\end{bmatrix}
+ \kappa(z, \lambda)
= \int_{-\lambda - \lambda'}^{\lambda - \lambda'}
\begin{bmatrix}
A & 3C & B & 3D \\
C & 3E & -D & 3F \\
B & -3D & A & -3C \\
D & 3F & -C & 3E
\end{bmatrix}
\begin{bmatrix}
\phi_0^+(z, \lambda) \\
\phi_1^+(z, \lambda) \\
\phi_0^-(z, \lambda) \\
\phi_1^-(z, \lambda)
\end{bmatrix}
d\lambda'
\]

\[
\sum_{p} \frac{Q_{p}}{\alpha(\lambda^{(p)}, \lambda)} \varrho(z, \lambda^{(p)}, \lambda)
\]
\[\alpha(\lambda^{(p)}, \lambda) \equiv \frac{\lambda^{(p)}}{4\pi} \frac{\left(\lambda^{(p)}\right)^2}{\mu_2^{(p)}}
\]

has been used, and the vector \( \varrho(z, \lambda^{(p)}, \lambda) \) has the expression

\[
\begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
+ \begin{bmatrix}
-B & 3D \\
D & 3F \\
0 & 1 \\
C & 3E
\end{bmatrix}
\begin{bmatrix}
V_{\rho^{(p)}(\mu^{(p)} | z)|} \\
V_{\rho^{(p)}(\mu^{(p)} | z)|} \\
V_{\rho^{(p)}(\mu^{(p)} | z)|} \\
V_{\rho^{(p)}(\mu^{(p)} | z)|}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & 3C \\
C & 3E \\
B & -3D \\
D & 3F
\end{bmatrix}
\begin{bmatrix}
V_{\rho^{II}(\mu^{II} | z)|} \\
V_{\rho^{II}(\mu^{II} | z)|} \\
V_{\rho^{II}(\mu^{II} | z)|} \\
V_{\rho^{II}(\mu^{II} | z)|}
\end{bmatrix}
\]

(35)

The coefficients A, B, C, D, E, F, which depend on \( \lambda - \lambda' \) (or \( \lambda - \lambda^{(p)} \)), are equal to sums of the type

\[
s(a, \beta) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{P_n(1+\lambda' - \lambda)}{P_n(1+\lambda' - \lambda)} a^n \beta^n
\]

in fact we have

\[
A = s(c_n^+, c_n^+), \quad B = s(c_n^+, c_n^-), \quad C = s(c_n^+, c_n^-),
\]

\[
D = s(c_n^+, c_n^-), \quad E = s(c_n^+, c_n^-), \quad F = s(c_n^+, c_n^-).
\]

They are further discussed in Appendix IV and in section 5.2. If (35) is premultiplied by the matrix \( -M \), where

\[
M =
\begin{bmatrix}
-3 & 3 & 0 & 0 \\
1 & -3 & 0 & 0 \\
0 & 0 & 3 & 3 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

(38)
the result is

\[
\frac{\partial}{\partial x} \begin{bmatrix}
\phi_0^+(x, \lambda) \\
\phi_1^+(x, \lambda) \\
\phi_0^-(x, \lambda) \\
\phi_1^-(x, \lambda)
\end{bmatrix} = \mu(x, \lambda) \begin{bmatrix}
-3 & 3 & 0 & 0 \\
1 & -3 & 0 & 0 \\
0 & 0 & 3 & 3 \\
0 & 0 & 1 & 3
\end{bmatrix} \begin{bmatrix}
\phi_0^+(x, \lambda) \\
\phi_1^+(x, \lambda) \\
\phi_0^-(x, \lambda) \\
\phi_1^-(x, \lambda)
\end{bmatrix} +
\int_{\lambda-2}^{\lambda} k(\lambda', \lambda) \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
c & d & a & b \\
g & h & e & f
\end{bmatrix} \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix} d\lambda'
\]

\[
\sum_{p} Q_{p} \delta(\lambda(p), \lambda) g(\lambda, \lambda(p), \lambda),
\]

where the vector \( \mathbf{s} = -M \mathbf{a} \) is equal to

\[
\begin{bmatrix}
3 \\
-1 \\
-3 \\
-1
\end{bmatrix} + \begin{bmatrix}
a & b & c & d \\
-3 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1
\end{bmatrix} \begin{bmatrix}
V_0(s_1(p), x) \\
V_1(s_1(p), x)
\end{bmatrix}
\]

and where \( a = 3A - 3C, b = 9C - 9E, c = 3B + 3D, d = 9D - 9F, e = -A + 3C, f = -3C + 9E, g = -B - 3D, \) and \( h = -3D + 9F. \) Defining

\[
\begin{bmatrix}
\phi_0^+(x, \lambda) \\
\phi_1^+(x, \lambda) \\
\phi_0^-(x, \lambda) \\
\phi_1^-(x, \lambda)
\end{bmatrix} = \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix}
\]

\[
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
c & d & a & b \\
g & h & e & f
\end{bmatrix} \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix} \int_{\lambda-2}^{\lambda} k(\lambda', \lambda) \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
c & d & a & b \\
g & h & e & f
\end{bmatrix} \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix} d\lambda'
\]

The result is

\[
\begin{bmatrix}
\phi_0^+(x, \lambda) \\
\phi_1^+(x, \lambda) \\
\phi_0^-(x, \lambda) \\
\phi_1^-(x, \lambda)
\end{bmatrix} = \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix} + \int_{\lambda-2}^{\lambda} k(\lambda', \lambda) \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
c & d & a & b \\
g & h & e & f
\end{bmatrix} \begin{bmatrix}
\phi_0^+(x, \lambda') \\
\phi_1^+(x, \lambda') \\
\phi_0^-(x, \lambda') \\
\phi_1^-(x, \lambda')
\end{bmatrix} d\lambda'+ \sum_{p} Q_{p} \delta(\lambda(p), \lambda) g(\lambda, \lambda(p), \lambda),
\]

(39) may be written in the condensed form:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \psi(x, \lambda) \\
\psi(x, \lambda)
\end{bmatrix} = \begin{bmatrix}
\mu(x, \lambda) M & \mu(x, \lambda) F \\
\mu(x, \lambda) J & \mu(x, \lambda) L
\end{bmatrix} \begin{bmatrix}
\psi(x, \lambda) \\
\psi(x, \lambda)
\end{bmatrix} +
\int_{\lambda-2}^{\lambda} k(\lambda', \lambda) \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
c & d & a & b \\
g & h & e & f
\end{bmatrix} \begin{bmatrix}
\psi(x, \lambda') \psi(x, \lambda') \\
\psi(x, \lambda') \psi(x, \lambda')
\end{bmatrix} d\lambda' + \sum_{p} Q_{p} \delta(\lambda(p), \lambda) \psi(x, \lambda(p), \lambda).
\]

3.2. Numerical Solution

In this section we shall consider the numerical solution of (39) or (43). The two variables in the problem, \( \lambda \) and \( x \), are treated differently: the wavelength is discretized and integration over \( \lambda \) is approximated by summation, whereas the integration of the equations with respect to \( z \) is carried out by a combination of an analytical and a least-squares method.

Let \( E_{\text{max}} \) be the highest energy in the source spectrum, and let the energy range of interest go down to some cut-off value \( E_{\text{cut}} \). A wavelength mesh is constructed by taking \( \lambda_{\text{min}} = f_{o}/E_{\text{max}} \) (\( f_{o} = 0.5110058 \)) and selecting a steplength \( \lambda \) such that \( 1/\lambda NE_{\text{max}} \) and \( \lambda_{\text{max}} \) the number of intervals, \( N \), are chosen such that \( \lambda_{\text{max}} = \lambda_{\text{min}} + N \lambda \) is approximately equal to \( f_{o}/E_{\text{cut}} \).
Solving the transport problem in terms of wavelength with a constant steplength \( \Delta \lambda = 1/m \) simplifies the calculations considerably, because the coefficients \( a, b, c, d, e, f, g \) and \( h \) in (39) can be calculated in advance for \( \lambda - \lambda' = 0, 1/m, 2/m, \ldots, 2-1/m, 2 \), before the \( \lambda \)-integration starts; this advantage would be lost if \( E \) were used instead of \( \lambda \). When the energy range is wide, however, neither \( E \) nor \( \lambda \) as integration variable yields a good economy of computation. Our interest is concentrated on the interval 0.1 - 2.6 Mev. If we choose \( \Delta \lambda = 1/64 \), say, and calculate \( \Delta E \) according to the formula

\[
\Delta E = - \frac{c}{a} \frac{d}{d\lambda} = - \frac{E^2}{f_0} \frac{d}{d\lambda},
\]

we find \( |\Delta E| \approx 0.2 \text{ Mev} \) at \( \lambda = 2.6 \text{ Mev} \) and \( |\Delta E| \approx 0.0003 \text{ Mev} \) at \( \lambda = 0.1 \text{ Mev} \). Hence, a constant steplength \( \Delta \lambda \) implies perhaps too large energy steps near the upper energy limit and unnecessarily small steps near the lower limit. The coarse energy-mesh width at 2.6 Mev seems tolerable because the scattered energy flux varies only slightly in this range (apart from the jumps from the source lines); but the very fine energy-mesh near 0.1 Mev involves indeed much computing time spent at low energies. For instance, a calculation from 2.6 Mev down to 0.1 Mev is approximately twice as expensive as the same calculation carried down to 0.2 Mev. An efficient strategy would be to divide the total range into subintervals and double the wavelength step from one subinterval to that next to it with lower energy. However, the price of this is a complication of the calculation scheme when describing transition of photons from one subinterval to another, and in view of the rather few calculations planned, we did not find it worthwhile.

In solving (43) a complication arises because the scattered flux, and hence the vector \( \mathbf{g}(z, \lambda) \) of expansion coefficients, has discontinuities induced by the source lines; line no. \( p \) causes one jump at \( \lambda = \lambda(p) \) and another at \( \lambda = \lambda(p) + 2 \) (fig. 2). Altogether 2P jumps may occur; at these jumps we shall calculate the magnitude of the jump

\[
\Delta \mathbf{g}(z, \lambda) = \mathbf{g}(z, \lambda+0) - \mathbf{g}(z, \lambda-0),
\]

as well as the limit from the left, \( \mathbf{g}(z, \lambda-0) \). It is readily seen that the jump magnitude for \( \lambda = \lambda(p) \) satisfies the equation

\[
\frac{3}{2} \frac{d}{d\lambda} \Delta \mathbf{g}(z, \lambda^0) = \mathbf{M} \cdot \Delta \mathbf{g}(z, \lambda^0)
\]

and for \( \lambda = \lambda(p) + 2 \) satisfies

\[
\frac{3}{2} \frac{d}{d\lambda} \Delta \mathbf{g}(z, \lambda^0 + 2) = \mathbf{M} \cdot \Delta \mathbf{g}(z, \lambda^0 + 2)
\]

The usual boundary conditions with respect to \( z \) apply to (46) and (47), i.e. \( \mathbf{g} \) must be continuous at \( z = 0 \) and finite in both limits \( z = \pm \infty \). Numerically, (46) and (47) are treated in complete analogy with (43).

Returning now to equation (43), its discrete counterpart with respect to wavelength is

\[
\frac{d}{dz} \psi_i(z) = \mathbf{M} \cdot \psi_i(z)
\]

\[
+ \sum_{j=i-1}^{i+1} \xi_{ij} k_{ij} P(i-j) \psi_j(z) + \sum_{P} Q_p \alpha(\lambda_p, \lambda_i) \delta(z, \lambda_i),
\]

Index \( i \) refers to the wavelength

\[
\lambda_i = \lambda_{\text{min}} + (i-1) \Delta \lambda \quad (1 \leq i \leq N+1).
\]

For brevity we have written \( \psi_i(z) \) for \( \psi_i(z, \lambda_i) \), \( \psi_j(z) \) for \( \psi_j(z, \lambda_j) \), and \( k_{ij} \) for \( k(\lambda_i, \lambda_j) \). The source term is obtained by moving each source wavelength \( \lambda(p) \) to its nearest wavelength mesh point \( i = j_p \) (several lines may fall in the same mesh point). The \( \xi_{ij} \) are suitably chosen quadrature weights (see Appendix V).

Appendix IV shows that for \( \lambda - \lambda' = 0 \) \( (i-j = 0) \) we have \( A = 1, \ B = 1, \ C = 1, \ D = 1, \ F = 1, \ T = 3, \ s = 3, \ t = 3, \ a = 1, \ b = 1, \ e = 1, \ f = 1, \ g = 1, \ h = 1, \) and \( \xi_{ij} = 1 \). This means that (48) is a system of the following type:

\[
\frac{d}{dz} \psi(z) = \mathbf{M} \cdot \psi(z) + \psi(z)
\]

\[
Q_p \alpha(\lambda_p, \lambda_i) \delta(z, \lambda_i),
\]

and for \( \lambda = \lambda(p) + 2 \) satisfies

\[
\frac{3}{2} \frac{d}{d\lambda} \Delta \mathbf{g}(z, \lambda^0 + 2) = \mathbf{M} \cdot \Delta \mathbf{g}(z, \lambda^0 + 2)
\]

\[
- Q_p \alpha(\lambda_p, \lambda_i) \delta(z, \lambda_i).
\]
with \( \Psi(z) = \Psi(z) - A \lambda_i \xi_{ii} \phi_i \) (index \( i \) in \( \phi \) and \( \xi \) is suppressed). The "source" \( \Psi(z) \) is the sum of a term due to real sources at \( \lambda = \lambda_i \) and slowing-down contributions from shorter wavelengths. \( \mu(z) \) can be regarded as an effective cross section, corrected for self-scattering in group \( i \).

Such a system exists for all the \( N+1 \) wavelength points in the range. It is now essential that we first solve (50) for the shortest wavelength \( (i = 1) \) and then for \( i = 2, i = 3, \ldots, i = N+1 \), in that order. In this way \( \Psi(z) \) will always be a known function. The structure of \( \Psi \) (cf. (38)) permits a partitioning of the vector equation (50) into one system for the up-going flux (the "plus component") and one for the down-going (the "minus component"):

\[
\frac{d}{dz} \begin{bmatrix} \psi^+(z) \\ \psi^-(z) \end{bmatrix} = \mu(z) \begin{bmatrix} \mathbb{I}^+ & 0 \\ 0 & \mathbb{I}^- \end{bmatrix} \begin{bmatrix} \psi^+(z) \\ \psi^-(z) \end{bmatrix} + \begin{bmatrix} \varphi^+(z) \\ \varphi^-(z) \end{bmatrix},
\]

i.e.

\[
\frac{d}{dz} \psi^\pm(z) = \mu^\pm(z) \psi^\pm(z) + \varphi^\pm(z),
\]

where

\[
\mathbb{I}^\pm = \begin{bmatrix} \mp 3 & 3 \\ 1 & \mp 3 \end{bmatrix}
\]

and

\[
\varphi^\pm(z) = \begin{bmatrix} \varphi_0^+(z) \\ \varphi_1^+(z) \end{bmatrix}
\]

(cf. (41); \( \varphi^+(z) \) has an expression analogous to (54)). We have now separated the plus and minus components of \( \Psi \), and they can be calculated by solving the two eqs. (52), which are of the common form

\[
\frac{d}{dz} \chi(z) = \mu(z) \mathbb{I} \cdot \chi(z) + \hat{g}(z).
\]

This vector equation represents two coupled linear differential equations. The standard way of solution is to apply a functional transformation

\[
\mathbb{I} = \mathbb{A} \chi
\]

to (55), such that the matrix of the new system becomes diagonal. Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of \( \mathbb{T} \). Then \( \mathbb{A} \) is selected as the matrix of eigenvectors of \( \mathbb{T} \), i.e.

\[
\mathbb{T} \mathbb{A} = \mathbb{A} \mathbb{D},
\]

where

\[
\mathbb{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\]

For \( \mathbb{T} = \mathbb{T}^+ \) we find

\[
\lambda_1^+ = \mp (3 + \sqrt{5}) \}
\lambda_2^+ = \mp (3 - \sqrt{5})
\]

and notice that

\[
\lambda_m < 0, \quad \lambda_m > 0 ;
\]

the corresponding eigenvector matrices are

\[
\mathbb{A} = \mathbb{A}^+ = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ \mp 1 & \mp 1 \end{bmatrix}
\]

The result of applying (56) to (55) is the system

\[
\frac{d}{dz} \chi(z) = \mu(z) \mathbb{D} \cdot \chi(z) + \hat{g}(z)
\]
with
\[ h = A^{-1} \cdot g \]  
and
\[ \Delta^{-1} = (\Delta^+)^{-1} = \begin{bmatrix} 1 & i \\ 273 & 1 \end{bmatrix} \]  

(63) splits into the two uncoupled scalar equations

\[ \frac{d}{dz} \chi_m(z) = \mu(z) A_m \chi_m(z) + h_m(z), \quad (m = 1, 2) \]  

(65)

We get a pair of equations like (67) for all four combinations of media and flux directions, i.e. (I, +), (I, -), (II, +) and (II, -):

\[ \chi^I_m(z) = \exp(\mu(z) A_m z) \left\{ \exp(-\mu(z) A_m z) h_m(z) dz + C_m \right\}, \quad (m = 1, 2) \]  

(68)

and

\[ \chi^I_m(z) = \exp(\mu(z) A_m z) \left\{ \exp(-\mu(z) A_m z) h_m(z) dz + C_m \right\}, \quad (m = 1, 2; z \leq 0) \]  

(69)

\[ \chi^I_m(z) = \exp(\mu(z) A_m z) \left\{ \exp(-\mu(z) A_m z) h_m(z) dz + C_m \right\}, \quad (m = 1, 2; z \geq 0) \]  

We get a pair of equations like (67) for all four combinations of media and flux directions, i.e. (I, +), (I, -), (II, +) and (II, -):

\[ \chi^I_m(z) = -\frac{I^I_m}{\mu^I A^I_m} + C_m \exp(\mu(z) A_m z) \sum_{j=1}^{h} \frac{I^I_m}{\mu_m z^j}, \quad (z < 0) \]  

(70)

and

\[ \chi^I_m(z) = -\frac{I^I_m}{\mu^I A^I_m} + C_m \exp(\mu(z) A_m z) \sum_{j=1}^{h} \frac{I^I_m}{\mu_m z^j}, \quad (z > 0) \]  

(71)

Here, \( \mu^I_m > 0 \) and \( \mu^I_m < 0 \), \( h^I_m \) and \( h^II_m \) must equal the limits \( h^I_m(-\infty) \) and \( h^II_m(\infty) \), which in turn are calculated from the vector \( g \) (cf. (55) and (63); as medium II is source-free, \( h^II_m = 0 \)).

The exact analytical representations of \( \chi_m(z) \) in (68) and (69) are complicated because the source term in our problem induces exponential integrals (cf. Appendix III), and the complexity increases rapidly as the wavelength integration proceeds. We choose instead an approximative method.

The physical nature of the problem indicates that each of the four \( h_m(z) \) can be adequately represented by a constant plus an exponential multiplied by a polynomial of degree \( k-1 \) in \( z \):

\[ h^I_m(z) = \mu^I_m \exp(\alpha^I_m z) y^I_m(z), \quad (m = 1, 2) \]  

(72)

and

\[ h^II_m(z) = \mu^II_m \exp(\alpha^II_m z) y^II_m(z), \quad (z > 0) \]  

(73)

where

\[ \mu^I_m = \frac{h^I_m}{a^I_m - \mu^I A^I_m}, \quad \mu^II_m = \frac{h^II_m - j \chi^II_m z^{j-1}}{a^II_m - \mu^II A^II_m} \quad (j = k-1, \ldots, 1) \]  

(74)

with an analogous definition of the \( \alpha^I_m \). It is supposed that the denominators in (74) are \( \neq 0 \). The case where one of them happens to be zero, or approximately zero, is handled as described in Appendix VII.
The integration constants $C_m^{I+}$ and $C_m^{II-}$ are determined from the boundary conditions:

\[ |\chi_m^{I+}(\infty)| < \infty, \quad \text{(75)} \]

\[ \chi_m^{I+}(0) = \chi_m^{II+}(3), \quad \text{(76)} \]

\[ |\chi_m^{II-}(\infty)| < \infty. \quad \text{(77)} \]

Remembering the signs of $A_m$ (see (60)) we derive from (75) that

\[ C_m^{I+} = 0 \quad \text{(78)} \]

and from (77) that

\[ C_m^{II-} = 0, \quad \text{(79)} \]

whereafter (76) yields

\[ C_m^{II+} = \kappa_m^{I+} - \kappa_m^{II+} - \frac{h_{m0} \kappa_m^{I+}}{\Lambda_m}. \quad \text{(80)} \]

And

\[ C_m^{I-} = \kappa_m^{II-} - \kappa_m^{I-} + \frac{h_{m0} \kappa_m^{II-}}{\Lambda_m}. \quad \text{(81)} \]

Now the transformed flux $\chi$ at the actual wavelength point can be calculated in our grid points ${\{z_j\}}$ as well as in prescribed calculation heights $\{h_n\}$ in medium II. The transformation back to $\phi$ is easily established (see (54) - (56) and (61)). The fluxes in $\{z_j\}$ are used for the subsequent flux calculations at longer wavelengths.

If medium II is a vacuum, the computational model outlined above requires certain modifications; these are described in Appendix VIII.

4. INTEGRAL FIELD QUANTITIES

In chapters 2 and 3 we discussed the calculation of the differential and angular energy flux $I(z, \omega, \lambda)$, which was the solution of the transport problem sketched in fig. 1. In this chapter we shall derive certain integral quantities of physical importance. We consider reference points in the source-free medium II, i.e. we assume $z \gg 0$.

When $I(z, \omega, \lambda)$ is integrated over $w$ we obtain the differential energy flux

\[ \phi(z, E) = \int_4^4 I(z, \omega, \lambda) d\Omega = \frac{2\pi}{\lambda} \int_1^1 I(z, \omega, \lambda) d\omega. \quad \text{(82)} \]

($E = f_0/\lambda, f_0 = 0.5110058$), and from $\phi$ we derive three integral field quantities by another integration from $E = E_1$ to $E = E_2$ (or from $\lambda = \lambda_2$ to $\lambda = \lambda_1$):$f_0/E_2$)

(i) The number flux (photons cm$^{-2}$ s$^{-1}$)

\[ N(z, E_1, E_2) = \int_{E_1}^{E_2} \phi(z, E)/E dE = \int_{E_1}^{E_2} \phi(z, E)/\lambda d\lambda. \quad \text{(83)} \]

(ii) The energy flux (Mev cm$^{-2}$ s$^{-1}$)

\[ N_e(z, E_1, E_2) = \int_{E_1}^{E_2} \phi(z, E)/\lambda d\lambda = \int_{E_1}^{E_2} \phi(z, E)/\lambda^2 d\lambda. \quad \text{(84)} \]

(iii) The absorbed dose rate (Mev g$^{-1}$ s$^{-1}$)

\[ D(z, E_1, E_2) = \int_{E_1}^{E_2} \phi(z, E)/\mu_e^\text{II}(E) dE = \int_{E_1}^{E_2} \phi(z, E)/\mu_e^\text{II}(E)/\lambda^2 d\lambda. \quad \text{(85)} \]

$\mu_e^\text{II}$ is the energy-absorption coefficient (cm$^2$ g$^{-1}$) for medium II.

When specializing eqs. (82) - (85) to the present problem, we partition all the quantities into uncollided and scattered components (as was the case for $I(z, \omega, \lambda)$ in eq. (5))

\[ \phi = \phi^{(u)} + \phi^{(s)}, \quad \text{(86)} \]

\[ N = N^{(u)} + N^{(s)} \quad \text{(87)} \]
\[ N_e = N_e^{(u)} + N_e^{(s)} \quad \text{(88)} \]

\[ D = D^{(u)} + D^{(s)} \quad \text{(89)} \]

and start to calculate the uncollided terms. We have

\[ \Phi^{(u)} = 2\pi \int_U(z, \omega, \lambda) d\omega \quad \text{(90)} \]

if the expression (10) for \( U \) when \( z > 0 \) is inserted, the result is

\[ \Phi^{(u)} = 2\pi \int_0^1 p(\lambda) \exp\left(-\frac{\lambda z}{\omega}\right) d\omega = 2\pi p(\lambda) E_2(\mu_\lambda^2 z) \quad \text{(91)} \]

where \( E_2 \) is the second-order exponential integral defined in Appendix III. By (11) this can also be written

\[ \Phi^{(u)} = \frac{E_2(\lambda) \mu_\lambda^2}{2} E_2(\mu_\lambda^2 z) \quad \text{(92)} \]

a well-known formula in radiation shielding.

When the line spectrum (33) is substituted for \( q(\lambda) \) we easily arrive at the following formulas:

\[ \Phi^{(u)} = \sum_{\gamma = 1}^p \frac{E_\gamma^H}{2\mu_\gamma^{(u)}} \delta(E - E_\gamma) E_2(\mu_\gamma^{(u)} z) \quad \text{(93)} \]

\[ N^{(u)} = \sum_{\gamma = 1}^p \frac{Q_\gamma^H}{2\mu_\gamma^{(u)}} E_2(\mu_\gamma^{(u)} z) \quad \text{(94)} \]

\[ N_e^{(u)} = \sum_{\gamma = 1}^p \frac{Q_\gamma E_\gamma}{2\mu_\gamma^{(u)}} E_2(\mu_\gamma^{(u)} z) \quad \text{(95)} \]

\[ D^{(u)} = \sum_{\gamma = 1}^p \frac{Q_\gamma E_\gamma}{2\mu_\gamma^{(u)}} \mu_\gamma^{(u)} E_2(\mu_\gamma^{(u)} z) \quad \text{(96)} \]

where the summations in (94) - (96) extend over the lines in the integration interval for \( E \).

Next we calculate the scattered components of (86) - (89). The scattered differential energy flux can be written (cf. (5), (12), (82)):

\[ \Phi^{(s)} = 2\pi \int_0^1 \psi(z, \omega, \lambda) d\omega \]

\[ = 2\pi \left[ \int_0^1 \psi(z, \omega, \lambda) d\omega + \int_0^1 \psi^*(z, \omega, \lambda) d\omega \right] \quad \text{(97)} \]

The "minus" and "plus" components \( \psi^- \) and \( \psi^+ \) are replaced by their double-P\(_1\) approximations (cf. (18))

\[ \psi^±(z, \omega, \lambda) \approx \psi_0^±(z, \lambda) + 3 \psi_1^±(z, \lambda)(2\omega - 1) \quad \text{(98)} \]

Only the zero'th harmonics contribute to (97) (cf. the orthogonality properties of \( P_\lambda \), Appendix I), and we find

\[ \Phi^{(s)} = 2\pi \left( \psi_0^+(z, \lambda) + \psi_0^-(z, \lambda) \right) \quad \text{(99)} \]

and from this follows immediately

\[ N^{(s)} = 2\pi \int_0^\lambda \left( \psi_0^+(z, \lambda) + \psi_0^-(z, \lambda) \right) d\lambda \quad \text{(100)} \]

\[ N_e^{(s)} = 2\pi \int_0^\lambda \left( \psi_0^+(z, \lambda) + \psi_0^-(z, \lambda) \right) d\lambda \quad \text{(101)} \]

and

\[ D^{(s)} = 2\pi \int_0^\lambda \left( \psi_0^+(z, \lambda) + \psi_0^-(z, \lambda) \right) \mu_\lambda^\Pi(E_\lambda) E_2(\mu_\lambda^2 z) d\lambda \quad \text{(102)} \]

5. DATA PROCESSING SYSTEM

A self-contained system of data files and computer programs related to the subjects treated in chapters 2-4 was established for operation on the B6700 at the Computer Installation at Risø. As the programs are written in FORTRAN and the data files contain card images, this data-processing system could be implemented on other machines without great effort.

The four data files are collected on a single multi-file magnetic tape GAMMA BANK and are described in detail in section 5.1.

The programs GAMP1 and GFX calculate double-P\(_1\) angular flux co-
coefficients and integral field quantities, respectively. Both use the GAMMA-BANK as data source, and GAMP1 may in addition pass calculation results to it. These programs are discussed in sections 5.1. and 5.2. In section 5.3. three editing programs GBUPDATE, GBCROS, and GBPRINT are discussed.

The operation of the complete data-processing system appears from the flow diagram in fig. 3.

5.1. Data File System: GAMMA BANK

The multi-file tape GAMMA BANK contains three files of basic data and one file of results and may be considered as the heart of the complete data-processing system, see fig. 3. This figure also shows the various possibilities for updating GAMMA BANK: the program GBUPDATE (sec. 5.4.) may update FILE1 and FILE2, whereas GAMP1 (sec. 5.2.) may update FILE4. Any updating of GAMMA BANK proceeds in the "ping-pong" mode with two magnetic tapes and with the disk of the computer as intermediate storage; at the next updating the two tapes are interchanged, and so on.

All the four files are composed of card images, i.e. the records are EBCDIC character strings of maximum length 80 char. In the following a short description is given of the contents of the files and the format of the records.

FILE1: Photon Emission Data

The photon source in our problem is the line spectrum from Th, U, or K. The evaluation of emission data for these radionuclides (number of photons per 100 disintegrations for each line) is discussed in Appendix XI. Here we shall only consider the representation of the data on FILE1. The structure appears from Appendix XII which contains a print-out per 1. Sep. 1974 of FILE1. The first record on the file is an identification no. for the whole tape (identifier NOTAPE, format 110). Each time GAMMA BANK is updated, NOTAPE is increased by 1. All remaining records in FILE1 have the form

ENERGY, YIELD, ILINE, NLINE, IEMIT

and the format F7.4, F7.2, 214, 16. The emitter code IEMIT is a four-digit number (pos. 25-28). The code for Th first digit is:

1: Thorium
2: Uranium
3: Potassium.

The last three digits form an isotope code, such that IEMIT altogether may assume the following four values:

1232 for $^{232}$Th
2238 for $^{238}$U
2235 for $^{235}$U
3040 for $^{40}$K.

All records with one emitter code are placed consecutively on the file. NLINE is the number of emission lines for the actual IEMIT. ILINE denotes the sequence no. of the actual line for the actual IEMIT, and ENERGY its energy. Increasing line nos. correspond to decreasing energies. YIELD denotes the intensity of the actual line, in photons per 100 disintegrations. The program GBUPDATE is used to perform updatings of FILE1, see section 5.4.

FILE2: Material Composition Data

A print-out of FILE2 as of 1st September 1974 is given in Appendix XII; it comprises the materials studied in chapter 6. A record has the form

IZ, WPCT, ICONST, NCONST, RHO, MIX

and the format 14, F9.4, 214, F12.6, 14. The composition code MIX lies in the interval 1 ≤ MIX ≤ 99 and characterizes the material. Records for one material are placed consecutively on the file. Their number is NCONST = number of elements in the material. RHO is the density of the material (g cm$^{-3}$). NCONST, RHO and MIX do not change for records belonging to one material; the first of these has ICONST = 1, the next ICONST = 2 and so on, until ICONST = NCONST. IZ is the atomic number; these values must be ranged in increasing order. WPCT is the weight percent of the actual elements in the material. The weight percents for one material must total 100.00 ± 0.01% to be accepted by the system. The program GBUPDATE is used to perform updatings of FILE2, see section 5.4.

FILE3: Cross Section Data

This file contains cross-section data in ENDF/B-format for the following 19 elements:
Cross sections are given for the following processes, characterized by the ENDF/B standard code MT:

<table>
<thead>
<tr>
<th>MT</th>
<th>TYPE OF CROSS SECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>501</td>
<td>Total</td>
</tr>
<tr>
<td>502</td>
<td>Coherent scattering</td>
</tr>
<tr>
<td>504</td>
<td>Incoherent scattering</td>
</tr>
<tr>
<td>516</td>
<td>Pair production (includes triplet)</td>
</tr>
<tr>
<td>602</td>
<td>Photoelectric</td>
</tr>
</tbody>
</table>

FILE3 was constructed by deletion of the complete Livermore library DCL-7D\(^{13}\) (this deletion was made by the U.S. program DAMMET). FILE3 cannot be updated by our data-processing system in fig. 3.

FILE4: Angular Flux Results

The records of FILE4 are produced by the double-\(P_1\) program GAMP1, cf. fig. 3 and section 5.2. Each record contains the 13 items:

\[
E, \lambda, \phi^+, \phi^-, \phi_0^+, \phi_0^-, \phi_1^+, \phi_1^-, i, d, N, z, IMIX1, IRADIO, IMIX2
\]

where:

- \(E\) = photon energy (MeV),
- \(\lambda\) = Compton wavelength,
- \(\phi^+, \phi^-, \phi_0^+, \phi_0^-, \phi_1^+, \phi_1^-\) = Expansion coefficients in the double-\(P_1\) approximation (eq. (98)) of the scattered flux (or the flux jump if \(d = 1\), see below),
- \(i\) = wavelength index (cf. (49)),
- \(d\) = discontinuity index:
  - \(d = 0\) if \(\lambda = \lambda_1\) is a point of continuity for \(\psi\),
  - \(d = 1\) if \(\lambda = \lambda_1\) is not a point of continuity for \(\psi\),
- \(N\) = number of wavelength intervals (cf. (49)),
- \(z\) = height (cm) of calculation point (fig. 1), \((z \geq 0)\)
- \(IMIX1\) = composition code for medium 1 (cf. FILE2),
- \(IRADIO\) = code for radio element in medium 1, being 1, 2, or 3 (cf. FILE1),
- \(IMIX2\) = composition code for medium 2 (cf. FILE2).

More details of these quantities and the organization of the records are given in section 5.2.

The format of a FILE4 record is:

\[
F6.4, F7.4, 1P4E11.4, 14, 11, 12
\]

A catalog of FILE4 per 1st September 1974 is given in Appendix XII.

5.2. The Program GAMP1

The FORTRAN program GAMP1 (Danish AEC program no. 648) carries out the double-\(P_1\) calculations outlined in chapter 3. Its position in the data processing system is apparent from fig. 3. The description given here refers to the version as of 1st September 1974.

Structure

GAMP1 consists of a driver program (MAIN), and the subprograms GAMPA, GAMP, DATAIO, MYG, AH, EXINT2, WQUADR, AKERNL, EXPPOL, COLDEC, and COLSOL. Variable dimensions are used in order to efficiently utilize the fast memory; the array bounds are passed from the
driver program to the master subroutine GAMPA (or GAMP if medium II is vacuum), which governs the flow of the calculations, see the flow diagram in fig. 4. The other subprograms are discussed below.

**Description of Subprograms**

**DATAIO** is a subroutine that reads input data from punch card, prints these data, and passes them to the main program.

**MYG** is a subroutine calculating a set of photon cross sections for a given set of energies, a given process type, and a given material. It applies double-logarithmic interpolation on the subset of Livermore tabulations in GAMMABANK/FILE3 (section 5.1). The material is specified by its MIX code (section 5.1); hence MYG can only calculate cross sections for those materials contained in GAMMABANK/FILE2. The process type is specified by the ENDF/B code MT, see section 5.1, FILE3.

**AH** is a subroutine that calculates the coefficients a, b, ..., h in (39). These numbers are linear combinations of A(Y), ..., F(Y) (see (39), (40) ff., and Appendix IV), which in turn are given as infinite series:

$$A(Y) = \sum_{n=0}^{\infty} \frac{2n+1}{2n} c_{2n}^2 P_n(Y) ,$$

(103)

and similar expressions for the others. They are to be calculated for a set of discrete and equidistant Y-values in the interval -1 ≤ Y ≤ 1 (as Y = 1 + λi - λ and λi = λ + λi + 2). From (A13)-(A18) it appeared sufficient to calculate directly only A(Y) and E(Y) and only for Y ≥ 0. In his paper, Gerstl operates with series of a nature similar to that of our series, also containing P n. He points out that the partial sum of such series,

$$\sum_{n=0}^{N} a_n ,$$

(104)

converges only slowly to the limit as N → ∞, but that s - s_N for large N fluctuates regularly around zero. In fact, the average

$$\overline{s_N} = \frac{1}{6} \sum_{m=1}^{6} s_{N-m+1} ,$$

(105)

has proved to approach s quite fast. In the present subroutine, \(s_{200}\) is used as an approximation to s; c_{no}, c_{nl}, and P_{n}(Y) are calculated by successive application of the appropriate recursion relations.

**EXINT2** is a function subroutine calculating the second-order exponential integral

$$E_2(x) = \int_{x}^{\infty} \frac{\exp(-t)}{t^2} dt$$

(106)

for x ≥ 0. The values obtained by EXINT2 were compared to Placzek's table reproduced in Goldstein pp. 358-65). In no case was a deviation found of more than one in the least significant decimal place.

**WQUAD** is a subroutine that calculates the quadrature weights in (A19) for arbitrary values m (= j_2 - j_1 + 1) of the number of quadrature points. The formulas are given in Appendix V.

**AKERNL** is a function subroutine which calculates the variable factor of the scattering kernel (2), viz.: 

$$k(\lambda^i, \lambda)/(\frac{3}{2} \Delta^\beta) = \frac{\lambda^i}{\lambda} \frac{\lambda^{i+1}}{\lambda} - 2(\lambda - \lambda^i) + (\lambda - \lambda^i)^2 .$$

(107)

**EXPPOL** is a subroutine that carries out the least-squares fitting discussed in Appendix VI.

**COLDEC** and **COLSOL** are subroutines to be used in the solution of systems of linear equations with positive-definite matrices, as they occur in EXPPOL. The two routines belong to the Danish AEC Library of FORTRAN Subprograms at Ris\\(\ddot{o}\) (SF/148).

**Specification of GAMPA Input Data**

GAMPA extracts information from the first three files of the GAMMABANK tape, but in addition it reads control parameters from punched cards according to the following prescript:
GAMP1 INPUT

<table>
<thead>
<tr>
<th>IDENTIFIER</th>
<th>FORMAT</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEADL</td>
<td>13A6, A2</td>
<td>Headline for problem</td>
</tr>
<tr>
<td>NOTAPE</td>
<td>I10</td>
<td>NOTAPEI must coincide with the identification no. for GAMMABANK before the updating. If NOTAPE&lt;0, the tape updating is suppressed, and only lineprinter output appears.</td>
</tr>
<tr>
<td>IMIX1, IRADIO, IMIX2</td>
<td>3110</td>
<td>IMIX1 is the material code for medium I. IRADIO denotes the radioelement in the following code: 1 for Th, 2 for U, 3 for K. IMIX2 is the material code for medium II. If this is a vacuum, set IMIX2=0.</td>
</tr>
<tr>
<td>ECUT</td>
<td>E10.0</td>
<td>Lower energy cutoff (Mev)</td>
</tr>
<tr>
<td>MLAM</td>
<td>I10</td>
<td>Number of intervals into which the unit wavelength is divided on calculating (m in fig. 2 and section 3.2); the standard value is 64.</td>
</tr>
<tr>
<td>conditioned NH on</td>
<td>I10</td>
<td>Number of calculation heights in medium II.</td>
</tr>
<tr>
<td>IMIX2 &gt; 0 (AZ\n(214,471),(273,490))</td>
<td>8E10.0</td>
<td>Calculation heights (cm).</td>
</tr>
</tbody>
</table>

In the present version of GAMP1 two parameters were fixed in the program itself:

NZ = 9. This is the n in Appendix VI denoting the number of fixed space mesh points in each medium.

KMI = 2. This is the degree of the polynomials used in least-squares fit (k - 1 in (70), (71)).

It was experience from the previous GAMP1 model that led to these choices.

Description of Output

Calculation results will normally be passed to the magnetic tape GAMMABANK, see section 5.2, FILE4. Each record transmitted to this file contains 13 items:

E, λ, ∗, +, +, +, •, i, d, N, z, IMIX1, IRADIO, IMIX2;

their formats and definitions were given in section 5.1. FILE4 is cumulative, so that records from earlier GAMP1 runs are saved.

Based on the GAMP1 input values ECUT and MLAM together with the maximum energy in the line spectrum of the actual radio element (FILE1), a set {λ₁} of equidistant wavelengths is constructed. A set {zₙ} of prescribed calculation heights was specified in the GAMP1 input. The results of the GAMP1 calculations are the double-P₁ expansion coefficients ∗(zₙ, λ₁) and +*(zₙ, λ₁), where 1 ≤ n ≤ N and 1 ≤ n ≤ n_max (the number of calculation heights). These double-indexed terms are calculated in the following order: first i = 1 and n = 1, 2, ..., n_max; then i = 2 and n = 1, 2, ..., n_max, and so on. However, it is more practical that the records on FILE4 are in the reverse order, i.e. all records for one height are consecutive. Hence the results must be reorganized before the transmission to tape, and this is done by means of the disk of the computer.

If we consider the variation of items in consecutive records on FILE4, then the triplet IMIX1/IRADIO/IMIX2 and N vary least rapidly and are constant within records belonging to one GAMP1 run. The next slowest variation has the height z. For each combination of the items above we have a cluster of records with fluxes at N different wavelengths {λ₁}. If λ = λ₁ is a continuity point for the flux, only one record exists with this λ and has d = 0, see description of FILE4 in section 5.1, but if λ = λ₁ is not a point of continuity, two records will be necessary; the first contains limits from the left, *(z, λ-0), etc., and has d = 0, the second contains the jumps *(z, λ-0) - *(z, λ-0), etc., and has d = 1.

The output transmitted to the tape will also appear on the lineprinter, and so will the input data.

The GAMP1 results are normalized to a radioelement content of 1 percent by weight of Th, U, or K. Together with the specific activities

4100 dis. per g Th per s,
12227 dis. per g U per s,
3,311 dis. per g K per s,

(see also Appendix XI), this fixes the source strength in photons/cm³/s.
Check Calculations, see Appendix IX and Appendix X.

5.3. The Program GFX

The FORTRAN program GFX (Danish AEC program no. 709) calculates integral field quantities on the basis of GAMP1 results, as outlined in chapter 4. Its position in the data processing system is apparent from fig. 3. The description given here refers to the version as of 1st September 1974.

Structure

GFX consists of a main program and the subprograms SORT, EXINT2, WQUADR, MYG, MYGEA, FINT.

The main program governs the flow of calculations and contains all the input/output instructions. It reads and prints out input data from cards (see Specification of GFX input) and reads source emission data from GAMMABANK/FILE1. After this, it calculates the uncollided contribution to the number flux, the energy flux, or the absorbed dose rate, whatever is desired, using (94), (95), or (96). To obtain the scattered flux or dose contribution, GFX reads data for the scattered angular flux in the specified height from FILE4, and evaluates the integral (100), (101) or (102). This evaluation is carried out by means of the quadrature formulas in Appendix V, with due attention to the jumps in the integrand and to fractional wavelength intervals at both limits.

This evaluation is carried out by means of the quadrature formulas in Appendix V, with due attention to the jumps in the integrand and to fractional wavelength intervals at both limits.

Description of subprograms

SORT is a subroutine which ranges the energies in the emission spectrum in increasing magnitude, before the cross-section routine MYG is called.

EXINT2, WQUADR, and MYG are the same as in GAMP1, see section 5.1.

MYGEA evaluates the energy-absorption coefficient \( \sigma_{\text{abs}} \), if a dose rate calculation is desired (cf. (85)). \( \sigma_{\text{abs}} \) is computed by means of the analytical expression given in Goldstein \( \text{[9]} \).

FINT is a function subprogram which evaluates the integrand of (100), (101), or (102).

Specification of GFX input

GFX extracts information from all four files of the GAMMABANK tape, but in addition it reads control parameters from punched cards according to the following prescript:

<table>
<thead>
<tr>
<th>IDENTIFIER</th>
<th>FORMAT</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic data</td>
<td>K</td>
<td>I10</td>
</tr>
<tr>
<td></td>
<td>ALPHA</td>
<td>A3</td>
</tr>
<tr>
<td></td>
<td>NINT</td>
<td>I10</td>
</tr>
<tr>
<td></td>
<td>(EMINI(I), EMAXI(I), 1 = 1, NINT)</td>
<td>16F5.0</td>
</tr>
<tr>
<td></td>
<td>IPUNCH</td>
<td>I10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Repetitive data</th>
<th>HEADL</th>
<th>13A6, A2</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMIX1, IRADIO, IMIX2</td>
<td>3110</td>
<td></td>
</tr>
<tr>
<td>NH</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>(AZH(N), N=1, NH)</td>
<td>8E10.0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Description</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic data</td>
<td>Calculation type</td>
</tr>
<tr>
<td></td>
<td>K=1: Number flux calculation</td>
</tr>
<tr>
<td></td>
<td>K=2: Energy flux calculation</td>
</tr>
<tr>
<td></td>
<td>K=3: Dose rate calculation</td>
</tr>
<tr>
<td></td>
<td>DIR: Contribution from uncollided radiation only</td>
</tr>
<tr>
<td></td>
<td>SCT: Contribution from scattered radiation only</td>
</tr>
<tr>
<td></td>
<td>TOT: Contribution from total radiation</td>
</tr>
<tr>
<td></td>
<td>Number of energy intervals in which flux or dose should be calculated</td>
</tr>
<tr>
<td></td>
<td>Energy limits for interval nos. 1, 2,..., NINT</td>
</tr>
<tr>
<td></td>
<td>IPUNCH = 0: Output on lineprinter only</td>
</tr>
<tr>
<td></td>
<td>IPUNCH = 1: Additional punched output (&quot;spectrum packages&quot;)</td>
</tr>
<tr>
<td></td>
<td>Headline for problem</td>
</tr>
<tr>
<td></td>
<td>IMIX1 is the material code for medium 1.</td>
</tr>
<tr>
<td></td>
<td>IRADIO denotes the radionelement in the following code: 1 for Th, 2 for U, 3 for K.</td>
</tr>
<tr>
<td></td>
<td>IMIX2 is the material code for medium II. If this is vacuum, set IMIX2 = 0</td>
</tr>
<tr>
<td></td>
<td>Number of calculation heights in medium II</td>
</tr>
<tr>
<td></td>
<td>Calculation heights (cm)</td>
</tr>
</tbody>
</table>

return to "Repetitive data" so long as all data cards have not been read.
Description of Output

The lineprinter output from GFX explains itself. As was the case for GAMP1, the GFX results are normalized to 1 percent Th, U, or K, by weight.

Check Calculation, see Appendix X.

5.4. Editing Programs

The three auxiliary programs GBUPDATE, GBCROS, and GBPRINT (fig. 3) carry out various editing tasks related to the files in GAMMABANK.

GBUPDATE (Danish AEC program no. 710) is used for updating FILE1 and FILE2. To update FILE1, a totally new card deck (excluding the tape identification no.) is provided to replace the old file. FILE2 may be extended by the addition of new materials. Card input is prepared according to the following scheme:

**GBUPDATE INPUT**

<table>
<thead>
<tr>
<th>IDENTIFIER</th>
<th>FORMAT</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>NUPDT1, NUPDT2</td>
<td>2110</td>
<td>NUPDT1 is the number of emitters in a total replacement of FILE1. NUPDT1 = 0: no replacement. NUPDT2 is the number of materials which are put on FILE2 from cards. NUPDT2 = 0: no updating.</td>
</tr>
<tr>
<td>NOTAPE</td>
<td>110</td>
<td>Identification no for GAMMABANK-tape before updating.</td>
</tr>
<tr>
<td>ENERGY, YIELD, ILINE, NLINE, IEMIT</td>
<td>F7.4, F7.2, 214, 16</td>
<td>Replacement cards to a new FILE1 (if NUPDT1 &gt; 0).</td>
</tr>
<tr>
<td>MAT, WPCT, ICONST, NCONST, RHO, IMIX</td>
<td>I4, F9.4, 214, F12.6, I4</td>
<td>Updating cards to FILE2 (if NUPDT2 &gt; 0).</td>
</tr>
</tbody>
</table>

The program carries out an extensive check of the updated information.

GBCROS (Danish AEC program no. 711) calculates gamma cross sections (cm⁻¹) for materials in the GAMMABANK by means of the subroutine MYG (section 5.2). For each material and each type of process one or more energies may be specified, see the input scheme below:

**GBCROS INPUT**

<table>
<thead>
<tr>
<th>IDENTIFIER</th>
<th>FORMAT</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>MIX, MT, NE</td>
<td>3110</td>
<td>MIX = material code. MT = ENDF/B code for process type: 501: total, 502: coherent scattering, 504: incoherent scattering, 516: pair production, 602: photoelectric effect. If total minus coherent is desired, put MT = 0. NE = 0 means that an equidistant set of energies is assumed. If NE &gt; 0 then NE = number of arbitrary input energies.</td>
</tr>
<tr>
<td>EMIN, EMAX, M</td>
<td>2E10.0, II0</td>
<td>Equidistant set of energies from EMIN to EMAX with spacing (EMAX-EMIN)/M.</td>
</tr>
<tr>
<td>E(I), I=1, NE</td>
<td>8E10.0</td>
<td>Photon energies ranged in increasing order.</td>
</tr>
<tr>
<td>IPUNCH</td>
<td>110</td>
<td>IPUNCH = 0: Output on lineprinter only. IPUNCH = 1: Additional punched output (&quot;spectrum packages&quot;).</td>
</tr>
</tbody>
</table>

return to beginning of scheme so long as all data cards have not been read.

GBPRINT (Danish AEC program no. 712) prints out the contents of the files of GAMMABANK, either as complete printouts or as catalogs.
Appendix XII was produced by means of GBPRINT.

6. RESULTS FOR Th-U-K GAMMA-RADIATION FIELDS IN WATER

A collection of tables and graphs has been prepared with the purpose of illustrating one possible application of the data-processing system described in chapter 5. For the configuration studied, the source material (medium I) is quartz sand saturated with water (bulk density = 1.88 g/cm³), and the source-free material (medium II) is water. We assume that the sand contains small traces of either Th, U, or K, but shall for convenience normalize all our results to a reference concentration of 1 percent radioelement. The radiation field is considered at the sand-water interface (z = 0) and at the distances z = 10, 20, 30, and 40 cm from the interface. These conditions were chosen for two reasons:

1. We know from an earlier investigation¹⁹) that the computational method is accurate to within 10% for determination of scalar flux densities in water up to z = 50 cm.

2. The results given are relevant to the interpretation of radiometric surveys of sea-bed formations.

We point out that the sand/water configuration is less special than one might believe. First, the radiation field in medium II depends little on the composition of medium I so long as the latter contains no elements with high atomic numbers. Secondly, if the distance z is measured in units of the mean free path in medium II, the field is fairly insensitive to variations in the composition of medium II. For example, the radiation field in air at z = 300 m is very similar to that in water at z = 40 cm.

In tables 2 through 16 we present the energy distribution of the scalar number flux of uncollided as well as scattered photons produced by each radioelement at the five reference levels in the water. As the flux is given at intervals as small as 0.01 Mev, the tables may be used for calculation of the corresponding pulse-height spectra of a gamma-ray detector with approximately constant pulse-height spectra of a gamma-ray detector with known response function¹⁹).

Table 17 shows the dose rate produced by the gamma-ray flux for each value of z. The dose rate was calculated as the sum of one term due to photons with E > 0.1 Mev and another term with E < 0.1; the latter contribution was obtained by extrapolation.

For a selected level in the water, z = 20 cm, the flux distributions are shown graphically in figs. 9, 10, and 11. It is instructive to compare the emission spectra of Th and U (figs. 7 and 8) with the flux distributions to which these radioelements give rise (figs. 10 and 11). The comparison clearly illustrates the two basic characteristics of photon transport phenomena: the attenuation of the uncollided flux components and the build-up of a scattered flux component.

Finally, we have studied the angular distribution of the photon flux in water above sand with potassium as the radioelement. Fig. 12 shows the distribution of uncollided 1.461 - Mev photons in the water at z = 10 cm and z = 40 cm, calculated from the analytical expression in chapter 2 (eq. (10)). We notice that the distribution is peaked in the upward direction w = 1, and that this peaking becomes more pronounced with increasing z. Fig. 13 shows the angular distribution of scattered photons for the same configuration, but for the photon energies 1.0 Mev, 0.3 Mev, and 0.1 Mev; these graphs were constructed from the expression for the double-P₁ approximation, eq. (98). The graphs clearly demonstrate the increasing amount of "skyshine" (photons with w < 0) when the energy decreases, and that the distributions for fixed energy tend to be more upward-peak with increasing height in medium II. The small jumps in the transition from positive to negative w reflect the truncation error in the double-P₁ approximation. This error is smallest when z is small, in agreement with the general experience that the accuracy of the double-P₁ method is best near the boundary¹⁷). The graphs presented in figs. 12 and 13 would be very similar to those obtained if medium II were air with z = 80 m and z = 300 m, respectively.
CONCLUSION

Starting with the double-$P_1$ approximation, the photon transport problem was solved for semi-infinite, plane-geometry conditions. The solution forms the basis of a data-processing system for computational evaluation of the natural radiation fields above plane geologic formations. As an application of the system, calculations were made of the contributions from thorium, uranium, and potassium to the radiation field in water superposing sand. The system appears to be very suitable as a source of information in computational studies of aerial, gamma-spectrometric survey techniques.

REFERENCES


2) H. Beck and G. de Planque, The Radiation Field in Air Due to Distributed Gamma-Ray Sources in the Ground. HASL-195 (1968) 59 pp.

3) H.L. Beck, J. De Campo, and C. Gogolak, In situ Ge (Li) and NaI(Tl) Gamma-Ray Spectrometry. HASL-258 (1972) 75 pp.


**APPENDIX I**

THE POLYNOMIALS $P^+_1(\omega)$

From the definition $P^+_1(\omega) = P_1(2\omega + 1) H(\omega)$ ($1 = 0, 1, 2, \ldots$) and the recurrence relation for $P_1(\omega)$,

$$\omega P_1(\omega) \cdot \frac{1 + 1}{21 + 1} P_{1+1}(\omega) + \frac{1}{21 + 1} P_{1-1}(\omega),$$

the following recurrence formula is obtained:

$$P^+_1(\omega) = 2 \omega P^+_1(\omega) - \frac{1}{21 + 1} P_{1+1}(\omega) - \frac{1}{21 + 1} P_{1-1}(\omega). \quad (A1)$$

The orthogonality relations are

$$\int P^+_1(\omega) P^+_m(\omega) d\omega = \frac{1}{21 + 1} \delta_{1m} \quad (A2)$$

**APPENDIX II**

THE COEFFICIENTS $c^+_n$

From the definition $c^+_n = \int P^+_n(\omega) P^+_1(\omega) d\omega$, some elementary properties are deduced:

$$c^+_n + c'^+_n = 2 b^+_n \quad (A3)$$

$$c^+_n = (-1)^{n+1} c^+_n \quad (A4)$$

hence only $c^+_n$ need be considered (the superscript + is dropped in the following):

$$\forall 1 \in \mathbb{N}: \quad c^+_n = 0 \quad (A5)$$

$$c^+_2 \omega, \omega \delta_{\omega, \omega} \quad (A6)$$

$$c^+_2 \omega, \omega = \frac{(-1)^{n-1}(2\omega)^2}{2^2 \omega(2\omega)(\omega!)} \quad (A7)$$
(cf. ref. 11 p. 306). (A6) and A7) give a convenient calculation procedure for $c_{no}$:

$$c_{oo} = 1, \quad c_{10} = \frac{1}{2}, \quad c_{no} = -\frac{2}{n+1} c_{n-2}, \quad \text{o, n} = 2, 3, 4, \ldots \tag{A8}$$

It may be shown that $c_{n1}$ can be calculated from

$$c_{n1} = \frac{2}{n+2} c_{n-1}, \quad \text{o, n} = 0 \quad \tag{A9}$$

Finally we shall prove that

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} c_{n1} = \frac{1}{2}\pi \quad \tag{A10}$$

We define a function $Y_1(w)$, $-1 < w < 1$:

$$Y_1(w) = \begin{cases} 0 & \text{if } w < -1 \\ 1 & \text{if } w > 1 \\ P_l(\omega) & \text{if } 0 < w < 1 \end{cases}$$

and expand it in spherical harmonics, $Y_1(w) = \sum_{n=0}^{\infty} a_{nl} P_n(\omega)$ with

$$a_{nl} = \frac{2n+1}{2} \int_0^1 P_l(\omega) P_n(\omega) d\omega = \frac{2n+1}{2} c_{n1}$$

But from (A2) we obtain

$$\frac{1}{2\pi} \int_{-1}^{1} \left| Y_1(w) \right|^2 d\omega = \sum_{n=0}^{\infty} a_{nl}^2 = \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{n1}^2$$

and thus (A10) is verified.

**APPENDIX III**

**THE INTEGRALS $V_1$**

The source-induced integrals $V_1(y) = \int_0^1 P_1(\omega) \exp(-\frac{y}{\omega}) d\omega$ can be expressed in terms of the second-order exponential integral

$$E_2(y) = y \int_{\gamma}^{\infty} \frac{\exp(-t)}{t^2} dt$$

For $1 = 0$ and $1$ we find

$$V_0(y) = E_2(y) \tag{A11}$$

$$V_1(y) = \exp(-y) - (1+y) E_2(y) \tag{A12}$$

**APPENDIX IV**

**THE COEFFICIENTS A, B, C, D, E, F**

These figures were defined after eq. (37); they are functions of $\lambda - \lambda'$, or, which is the same, of the parameter $\gamma = 1 + \lambda' - \lambda$. It is easily shown that there are the following relations between them:

$$A(1) + B(\gamma) = 1 \quad \tag{A13}$$

$$C(\gamma) - D(\gamma) = 0 \quad \tag{A14}$$

$$E(\gamma) + F(\gamma) = \frac{1}{2} \gamma - 2C(\gamma) \quad \tag{A15}$$

$$A(\gamma) + C(\gamma) = \frac{1}{2} + \frac{1}{2} \gamma \quad \tag{A16}$$

Once $A$ and $E$ are calculated, the others follow from these formulas. Further, it is only necessary to compute $A(\gamma)$ and $E(\gamma)$ for $\gamma = 0$ owing to the relations

$$A(-\gamma) = 1 - A(\gamma) \quad \tag{A17}$$

$$E(-\gamma) = 1 + \frac{1}{2} \gamma - 2A(\gamma) + E(\gamma) \quad \tag{A18}$$

For $\gamma = 1$ we find $A(1) = \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{n0}^2$ and $E(1) = \sum_{n=0}^{\infty} \frac{2n+1}{2} c_{n1}^2$

(A10) gives immediately $A(1) = 1$ and $E(1) = \frac{1}{2}$, whereas (A13-16) give $B(1) = 0$, $C(1) = 0$, $D(1) = 0$, and $F(1) = 0$. 


APPENDIX V
NUMERICAL QUADRATURE

Eq. (43) contains integrals which in (48) were replaced by sums

\[
\int_{\lambda_{1}}^{\lambda_{2}} F(\lambda) d\lambda \approx \sum_{j=1}^{j_{2}} w_{j} F(\lambda_{j}) \Delta \lambda ; \quad (A19)
\]

we shall in particular be concerned with the quadrature weights \( w_{j} \). The integrand may have discontinuities in the interval considered; such points will be taken as boundaries between different quadrature ranges for evaluation of the sum, which breaks into parts representing intervals of continuity (for a discontinuity point, \( w_{j} \) will be the sum of two terms).

In (A19) we may hereafter suppose that \( F(\lambda) \) is continuous, and we shall state the applied quadrature rules with the associated weights \( w_{j} \). The formula chosen depends on the number of intervals \( n = j_{2} - j_{1} \), as specified below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Quadrature rule</th>
<th>Quadrature weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trapezoidal</td>
<td>( w_{j} = w_{j+1} = \frac{1}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>Cote's 3rd order (^{12})</td>
<td>( w_{j} = w_{j+1} = \frac{3}{8}, \quad w_{j+2} = \frac{3}{8} )</td>
</tr>
<tr>
<td>even</td>
<td>Simpson</td>
<td>( w_{j} = w_{j+1} = \frac{1}{3}, \quad w_{j+2} = \frac{2}{3} )</td>
</tr>
<tr>
<td>odd, 3</td>
<td>Cote's 3rd order for the short-wavelength part, Simpson for the remaining</td>
<td>Combination of the two types above</td>
</tr>
</tbody>
</table>

The transformation parameter \( c \) is taken to be equal to the \( \gamma \)-ray cross section for the medium at the shortest wavelength in the calculation range (this choice was promoted by considerations of numerical stability).

Each \( h(z) \) of (A20) is well-defined (although it might have resulted from fitting procedures at shorter wavelengths), so the values \( h(z_{i}) \) can be calculated. The constant terms in (70) and (71) could be found analytically. Hence the general problem we face is to find a set of parameters \( \{ a_{1}, \ldots, a_{k}, \beta \} \) that makes the function

\[
f(x) = \sum_{j=1}^{k} \alpha_{j} \varphi_{j}(x; \beta)
\]

with \( \varphi_{j}(x; \beta) \equiv \exp(-c x) x^{j-1} \)

at \( \{ x_{1}, \ldots, x_{n} \} \) take on values \( \{ f_{1}, \ldots, f_{n} \} \) that are as close as possible to prescribed ordinates \( \{ y_{1}, \ldots, y_{n} \} \). A least squares fit is obtained by requiring that

\[
\Phi = \sum_{i=1}^{n} w_{i}(y_{i} - f_{i})^{2} \quad (A23)
\]

must be minimum. \( w_{i} \) are the weights of the data points (in our case, the choice \( w_{i} = 1 \) (cf. (A21)) was made; this emphasizes a good fit at great heights in medium II). We use the semi-linear method described in ref. 10. In this method, Marquardt iterations are performed in the non-linear space, which in our case is the one-dimensional \( \beta \)-space. The linear space is \( k \)-dimensional (cf. (70) and (71)) with points \( \mathbf{g}^{T} = (a_{1}, \ldots, a_{k})^{T} \). The controlling equation \( (A + A^{T}) \mathbf{g} = 10 \) for Marquardt iterations specializes to

APPENDIX VI
LEAST-SQUARES FIT

We shall consider the determination of the coefficients in the expressions (70) and (71) for each function

\[
h(z) = \begin{cases} \frac{1}{h_{m}(z)} & z < z_{i} \\ \frac{1}{h_{m}(z)} & z > z_{i} \end{cases} \quad (A20)
\]

In each medium a fixed set \( \{ z_{i} \} \) of discrete \( z \)-values \( (i = 1, \ldots, n) \) is selected in advance; we have chosen this set such that

\[
ex(-c z_{i}) = 1 - (i-1)/n . \quad (A21)
\]

The transformation parameter \( c \) is taken to be equal to the \( \gamma \)-ray cross section for the medium at the shortest wavelength in the calculation range (this choice was promoted by considerations of numerical stability).

Each \( h(z) \) of (A20) is well-defined (although it might have resulted from fitting procedures at shorter wavelengths), so the values \( h(z_{i}) \) can be calculated. The constant terms in (70) and (71) could be found analytically. Hence the general problem we face is to find a set of parameters \( \{ a_{1}, \ldots, a_{k}, \beta \} \) that makes the function

\[
f(x) = \sum_{j=1}^{k} \alpha_{j} \varphi_{j}(x; \beta)
\]

with \( \varphi_{j}(x; \beta) \equiv \exp(-c x) x^{j-1} \)

at \( \{ x_{1}, \ldots, x_{n} \} \) take on values \( \{ f_{1}, \ldots, f_{n} \} \) that are as close as possible to prescribed ordinates \( \{ y_{1}, \ldots, y_{n} \} \). A least squares fit is obtained by requiring that

\[
\Phi = \sum_{i=1}^{n} w_{i}(y_{i} - f_{i})^{2} \quad (A23)
\]

must be minimum. \( w_{i} \) are the weights of the data points (in our case, the choice \( w_{i} = 1 \) (cf. (A21)) was made; this emphasizes a good fit at great heights in medium II). We use the semi-linear method described in ref. 10. In this method, Marquardt iterations are performed in the non-linear space, which in our case is the one-dimensional \( \beta \)-space. The linear space is \( k \)-dimensional (cf. (70) and (71)) with points \( \mathbf{g}^{T} = (a_{1}, \ldots, a_{k})^{T} \). The controlling equation \( (A + A^{T}) \mathbf{g} = 10 \) for Marquardt iterations specializes to
with \( a = \frac{w_i}{\varphi_i} \) and \( g = \sum w_i (y_i - f_1) \frac{\partial f_i}{\partial \beta} \). Given a \( \beta \) (guessed or iterated), the linear part of the problem will be to find the \( g \)-vector that minimizes \( \phi \). This \( g \) is a solution to the \( k \) th order linear system

\[
C \alpha = \gamma
\]

with

\[
C_{jk} = \sum_i w_i \varphi_{ij} \varphi_{kj} = \sum_i w_i \exp(\beta x_i) x_i^{-1} \exp(\beta x_i) x_i^{-1}
\]

and

\[
\gamma_j = \sum_i w_i y_i \varphi_{ij} = \sum_i w_i y_i \exp(\beta x_i) x_i^{-1}
\]

(F. \( \equiv \varphi(x_i; \beta) \)).

The derivative \( \frac{\partial f_i}{\partial \beta} \) to be used to form (A24) has the value

\[
\frac{\partial f_i}{\partial \beta} = \sum_j \left[ \frac{\partial \varphi_i}{\partial \beta} \varphi_{ij} + \alpha_j \frac{\partial \varphi_{ij}}{\partial \beta} \right] = \sum_j \left[ \frac{\partial \varphi_i}{\partial \beta} \exp(\beta x_i) x_i^{-1} + \alpha_j \varphi_{ij} \exp(\beta x_i) x_i^{-1} \right]
\]

\[= x_i f_i + \exp(\beta x_i) \sum_j \frac{\partial \varphi_i}{\partial \beta} x_i^{-1},\]

where \( \frac{d}{\partial \beta} \) satisfies the linear system

\[
C \theta = t
\]

with

\[
t_j = \sum_i w_i \left[ (y_i - f_1) \frac{\partial \varphi_i}{\partial \beta} - \varphi_{ij} \sum_{k=1}^k \frac{\partial \varphi_{ik}}{\partial \beta} \right]
\]

\[= \sum_i w_i (y_i - 2f_1) \exp(\beta x_i) x_i^{-1}.
\]

In the fitting procedure it was necessary to put some limitations on the variation of \( \beta \). First, the sign of \( \beta \) is fixed by the requirement that the exponential must decay when moving away from the interface; further, we prevent \( \beta \) from approaching zero or infinity by stating a lower and an upper limitation for \( \beta \), i.e., \( \beta_{\min} \leq |\beta| \leq \beta_{\max} \).

### APPENDIX VII

**INDETERMINATE X-EXPRESSIONS**

If one of the decay constants \( k_m^+ \) or \( k_m^- \), resulting from the least-squares fit and entering the expressions (70) and (71), happens to be equal to \( k_m^+ \) or \( k_m^- \), respectively, then the \( x \)-expressions (74) (or the analogous expressions for medium II) are indeterminate. In this case the solution (72) (or 73) must be replaced by

\[
\chi_m^+ (x) = - \frac{I_m^+}{\mu_m A_m} + C_m \exp(\mu_m A_m x) x \exp(\sigma_m x) \sum w_m j^m \frac{x}{x_0} + \sum w_m j^m \frac{x}{x_0} + \sum w_m j^m \frac{x}{x_0}
\]

\[(x < 0)
\]

with

\[
w_m = \frac{I_m^+}{\mu_m A_m}
\]

(or the analogous expression for medium II). The only differences between (A26) and (72) are that a factor \( z \) has entered into the last term in (A26) and that the \( w_m \) have different meanings. If this modification is necessary, we must also replace the corresponding \( w_m \) in the boundary conditions (80) and (81) by zero.

Numerical considerations lead to use of the modified procedure also when a denominator in (74) should be close to zero.

### APPENDIX VIII

**VACUUM IN MEDIUM II**

In this particular case the angular photon flux is constant everywhere in medium II, and we need only consider the transport equation in medium I (\( z > 0 \)). All the equations for this zone derived for non-vacuum medium II are still valid up to eq. (72), where we must replace the boundary conditions by

\[
\chi_m^+ (\infty) < \infty
\]

and

\[
\chi_m^- (0) = 0
\]

(A28) reflects the fact that we have no down-streaming photons at the interface. These conditions determine \( C_{k_m}^{+/-} \) in (72) to be:
An analytical check of the double-$P_1$ expansion coefficients calculated by GAMP1 is possible, if we consider photons from a single-line source in medium I that have suffered only glancing collisions. The energy flux of these photons satisfies (7), which in this particular case reduces to:

$$
\psi(z, \omega) = C_n(z) \psi(z, \omega).
$$

(A31)

The argument $\lambda = \lambda_0 f \omega / E_0$ (the source wavelength) has been dropped in (A31). The constant $C$ has the value

$$
C = \frac{\lambda_0 r_0 Q}{2 \mu_1},
$$

(A32)

where $r_0$ is the classical electron radius (see (2) and fig.), $Q$ is the source strength (photons/cm$^3$/s), and $n_1$ is the cross section for medium I at the source energy $E_0$. $n_e(z)$ is the density of electrons. As only positive values of $\omega$ are relevant in (A31), we have (cf. (10)):

$$
u_0(z, \omega) = \begin{cases} 1 & \text{for } z \leq 0 \\ \exp(-\frac{\mu_1}{\omega} z) & \text{for } z \geq 0 \end{cases}
$$

(A33)

($\sigma_II$ is the cross section for medium II at $E_0$). It is now easy to establish the complete solution of (A31) for $z \leq 0$ (medium I) as well as for $z > 0$ (medium II):

$$
\psi^+(z, \omega) = \frac{C n_e}{\mu_1} + A_1(\omega) \exp(-\frac{\mu_1}{\omega} z), \quad z \leq 0
$$

(A34)

$$
\psi^-(z, \omega) = \frac{C n_e}{\mu_1} z \exp(-\frac{\mu_1}{\omega} z) + A_II(\omega) \exp(-\frac{\mu_1}{\omega} z), \quad z > 0
$$

(A35)

($n_e^I$ and $n_e^{II}$ are the electron concentrations in the media; $A_1(\omega)$ and $A_II(\omega)$ are integration constants). The usual boundary conditions imply that $A_1(\omega) = 0$ and $A_II(\omega) = C n_e^I / \mu_1^I$, i.e. the glancing-scattered energy flux in medium II has the analytical expression:

$$
\psi(z, \omega) = C (\frac{n_e}{\omega} z + \frac{n_e}{\mu_1^I}) \exp(-\frac{\mu_1}{\omega} z).
$$

(A36)

($\omega > 0$, $z > 0$)

This flux can be represented exactly by an infinite expansion in half-range spherical harmonics:

$$
\psi^+(z, \omega) = \sum_{\ell=0}^{\infty} (2\ell + 1) \psi^+_{\ell}(z) P^+_{\ell}(\omega)
$$

(A37)

with expansion coefficients

$$
\psi^+_{\ell}(z) = C \left[ \int_{0}^{1} \left( \frac{n_e}{\omega} z + \frac{n_e}{\mu_1^I} \right) \exp(-\frac{\mu_1}{\omega} z) P_{\ell}(2\omega - 1) d\omega \right].
$$

(A38)

In particular we find for $\ell = 0$ and $\ell = 1$ that:

$$
\psi^+_0(z) = C \left[ n_e z E_1(\mu_1 z) + \frac{n_e}{\mu_1^I} E_1(\mu_1 z) \right]
$$

(A39)

and

$$
\psi^+_1(z) = C \left[ -n_e z E_1(\mu_1 z) + (2n_e z - \frac{n_e}{\mu_1^I}) E_1(\mu_1 z) + \frac{2n_e}{\mu_1^I} E_1(\mu_1 z) \right].
$$

(A40)
E_n(x) stands for the nth-order exponential integral (cf. Appendix III):

\[
E_n(x) = \int_0^1 e^{-x/\omega} \frac{d\omega}{\omega} \sum_{t=0}^{n-2} \omega^{-n-2} \exp\left(\frac{-x}{\omega}\right) d\omega
\]

Now, a comparison can be made between the analytical expressions (A39, A40) for the expansion coefficients \(v_0^Q\) and \(V_1^Q\), and the corresponding values calculated by GAMP1. The latter are approximate both because of the truncation of the double-\(P_1\) equations (20) beyond the first-order terms and because of the least-squares fitting in the variable \(z\). This comparison was made for water above sand with 1% K, and the result is shown in fig. 5 for the range of heights 0 \(\leq z \leq 40\) cm.

**APPENDIX X
MONTE CARLO CHECK OF GAMP1/GFX**

An independent check of the programs and files in our data processing system (fig. 3) was made by calculation of the absorbed dose rate in water above sand by GFX as well as by the Monte Carlo code MC4. The latter is primarily intended to solve shielding problems with multi-layer slabs of finite thicknesses. When applying it to the present problem, we assumed a 35 cm sand layer with uniform and monoenergetic K-sources and above this 100 cm water. Dose rates from MC4 throughout the range 0 \(\leq z \leq 60\) cm are given in fig. 6 as a histogram plot, and the corresponding GFX results as a curve drawn through five calculated ordinates. The effect of the finite extension of the media in MC4 will be small in the range considered. The computational principles underlying GAMP1/GFX and MC4 are of course completely different, but also the data sources differ: GAMP1/GFX uses point cross sections derived from the Livermore Library, whereas MC4 uses group-averaged cross sections based on an earlier compilation. Further, the sampling technique for picking deflection angles and energy losses in MC4 is an approximate device of Carlson. Taking all this into consideration, the agreement between the two models is satisfactory.

**APPENDIX XI
PHOTON EMISSION DATA**

The photon energies and the photon yields entering GAMMABANK/FILE1, and the specific activities used in the programs GAMM1 and GFX, were taken from references 15-18. Reference 15 is an up-to-date tabulation of emission lines from \(^{232}\)Th and \(^{238}\)U in secular equilibrium with their respective daughters. Since the photon emission spectrum of \(^{235}\)U + daughters is not reported in reference 15, this spectrum was evaluated independently from the decay schemes (\(^{235}\)U through \(^{207}\)Pb) given in reference 16. The specific activities of \(^{232}\)Th, \(^{238}\)U and \(^{235}\)U in natural thorium and uranium were chosen in accordance with the "best values" recommended in reference 17.

Data on the 1.461-Mev photon emission from \(^{40}\)K were derived from reference 18. The specific gamma activity of potassium was evaluated from table IX in reference 18, which summarizes 19 determinations carried out in the years 1950-1966. Only the 12 determinations for which an experimental error is stated are included. From a statistically weighted average of these we arrived at a figure of 3.31 \(\pm 0.04\) photons \(\cdot s^{-1} / g\) K.

A survey of the photon emission data adopted in this work is given in Table 1. It follows from the table that one gram of natural thorium and uranium results in the emission of 17370 and 33280 photons \(\cdot s^{-1}\) \(\cdot g^{-1}\) respectively. The emission spectra of thorium and uranium are shown in figs. 7 and 8. The average photon energies of the two radionuclides equal 0.591 and 0.617 Mev respectively.
Scalar number flux in the energy intervals 0.10 - 0.11, 0.11 - 0.12, ..., 2.99 - 3.00 MeV at different levels z in water above sand with a reference content of 1 percent Th, U, or K. The flux is due to uncollided as well as scattered photons. The unit is photons cm$^{-2}$ s$^{-1}$.

### Table 1

<table>
<thead>
<tr>
<th>Radioelement</th>
<th>No. of Y-emitters</th>
<th>Total photon emission lines</th>
<th>Spec. activity (dis./s.$^{-1}$ g radioactivity$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thorium</td>
<td>232Th-daughters</td>
<td>94</td>
<td>423.6</td>
</tr>
<tr>
<td>Uranium</td>
<td>238U-daughters</td>
<td>75</td>
<td>263.6</td>
</tr>
<tr>
<td>Potassium</td>
<td>39K</td>
<td>22</td>
<td>187.3</td>
</tr>
</tbody>
</table>
### Table 2

<table>
<thead>
<tr>
<th>MEV</th>
<th>0.00*0.01</th>
<th>0.01*0.02</th>
<th>0.02*0.03</th>
<th>0.03*0.04</th>
<th>0.04*0.05</th>
<th>0.05*0.06</th>
<th>0.06*0.07</th>
<th>0.07*0.08</th>
<th>0.08*0.09</th>
<th>0.09*0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>1.02% 0.01</td>
<td>2.04% 0.02</td>
<td>3.06% 0.03</td>
<td>4.08% 0.04</td>
<td>5.10% 0.05</td>
<td>6.12% 0.06</td>
<td>7.14% 0.07</td>
<td>8.16% 0.08</td>
<td>9.18% 0.09</td>
<td>10.20% 0.10</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>MEV</th>
<th>0.00*0.01</th>
<th>0.01*0.02</th>
<th>0.02*0.03</th>
<th>0.03*0.04</th>
<th>0.04*0.05</th>
<th>0.05*0.06</th>
<th>0.06*0.07</th>
<th>0.07*0.08</th>
<th>0.08*0.09</th>
<th>0.09*0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>1.02% 0.01</td>
<td>2.04% 0.02</td>
<td>3.06% 0.03</td>
<td>4.08% 0.04</td>
<td>5.10% 0.05</td>
<td>6.12% 0.06</td>
<td>7.14% 0.07</td>
<td>8.16% 0.08</td>
<td>9.18% 0.09</td>
<td>10.20% 0.10</td>
</tr>
</tbody>
</table>
### Table 4

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### Table 6

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### Table 7

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### Table 8

<table>
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### Table 9

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### Table 12

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<td>0.18</td>
<td>0.004±0.01</td>
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<tr>
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<td>0.055±0.02</td>
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<td>0.40</td>
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### Table 13

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<tr>
<td>Neve · g' · z· 1 = &amp;7,678 rad/h</td>
<td>0.00187</td>
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</table>

Dose rates in water at the distance z from the source material (cm)

\[ K = \frac{\text{dose rate (MeV · g' · cm)}^{-1}}{\text{per} \% \text{of radionuclide}} \]

Table 17

\[ K \cdot z = 40 \text{ cm} \]

Table 16
Fig. 1. Geometry for the Two-media Problem.

Fig. 2. Wavelength Scale and a Qualitative Solution for a Scattered Flux Component.

\[ \cos \Theta = \omega \]

**Expansion coefficient** \( \Psi^2(z, \lambda) \)

for scattered flux

- \( \lambda_{\text{min}} \approx f_0 / E_{\text{max}}, \lambda_{\text{max}} \approx f_0 / E_{\text{cut}} \)
- \( N = (\lambda_{\text{max}} - \lambda_{\text{min}}) / \Delta \lambda, m = 1 / \Delta \lambda \)

In this example, \( N = 24 \) and \( m = 4 \)
Fig. 3. Flow Diagram for Complete Data Processing System.

Fig. 4. Flow Diagram for GAMP1/SEP74.
Fig. 5. Expansion Coefficients for Scattered Flux at Source Wavelength.

Fig. 6. Absorbed Dose Rate in Water above Sand.
Fig. 7. Photon Emission Spectrum of $^{\text{232}}\text{Th}$ in Secular Equilibrium.

Fig. 8. Photon Emission Spectrum of $^{\text{238}}\text{U} + ^{\text{235}}\text{J}$ in Secular Equilibrium.
Fig. 9. Energy Distribution of the Scalar Photon Flux in Water Produced by Th in Sand.

Fig. 10. Energy Distribution of the Scalar Photon Flux in Water Produced by U in Sand.
Fig. 11. Energy Distribution of the Scalar Photon Flux in Water Produced by K in Sand.

Fig. 12. Angular distributions (relative flux per steradian) of uncollided 1.46 MeV photons in water. $\theta = 0^\circ$ corresponds to top orientation.
Fig. 13. Angular Distributions (relative flux per steradian) of scattered photons in water at selected heights and energies, calculated by the double-$P_1$ approximation. The source is $^{40}$K in the underlying sand. $\theta = 0^\circ$ corresponds to top orientation.