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Strong Turbulence in Magnetized Plasmas

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Abstract

By means of a fluid description, an investigation is made of the spectral structure of turbulence in a plasma confined by a strong homogeneous magnetic field. The turbulent spectrum is divided into subranges. Mean gradients of velocity and density excite turbulent motions, and serve as sources in the production subrange. The spectra of velocity and potential fluctuations interact in the coupling subrange, and the energy is transferred

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along the spectrum in the inertia subrange, to be ultimately dissipated in a dissipation subrange. Applying the method of cascade decomposition, we obtain the spectral laws k^{-3}, k^{-3}, k^{-2} for the velocity fluctuations, and k^{-3}, k^{-5}, $k^{-3/2}$ for the potential fluctuations, in the production, coupling and inertia subranges, respectively. The Bohm diffusion coefficient is also reproduced. Good agreement is found with measured power laws reported in the literature.

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1. INTRODUCTION

It is now generally accepted that the macroscopic transport properties of partially or fully ionized plasmas may be considerably modified if the intensity of the fluctuations associated with the collective degrees of freedom greatly exceeds the thermal equilibrium value. These collective or "anomalous" properties of turbulent plasmas are particularly important to fusion research because they control the plasma confinement. The subject is also of great interest to other branches of plasma physics, e.g. astrophysics.

In the present work we are concerned with low frequency oscillations in a non-uniform plasma confined by a strong, homogeneous magnetic field. Such a configuration is known to favour (linearly) unstable oscillations of the drift-wave type (see e.g. refs. 1-5). For the purpose of investigating the spectra of potential and velocity fluctuations, we apply the method of cascade decomposition[6)]. This choice is not necessarily self-evident; one could apply an alternative method based on resonant wave-wave and wave-particle interaction. Such a method has been highly successful in describing certain phenomena in weak plasma turbulence[4,7)], e.g. the well known Kadomtsev spectrum, but it is not valid in strong turbulence[4,8)]. This is by no means surprising, as wave-wave interaction may be visualized as quasi-particle interaction with conservation of energy of momentum. When strong turbulence sets in, the identification of a wave as a quasi-particle is no longer meaningful. In this case it is more likely that the energy is cascaded along the spectrum in wave vector space, in a manner very similar to that applied in the dimensional analysis by Kolmogorov and Obukhov in their investigations of fluid turbulence (for a review, see Chandrasekhar, ref. 9). Because we expect drift waves to induce strongly turbulent fluctuations, we apply the method of cascade decomposition. We may mention, however, that some authors attempt to evade the difficulties of the wave-wave interaction description in strong turbulence by introducing a new elementary excitation; for example, by describing strong Langmuir turbulence by a superposition of self-trapped plasma waves or solitons[10)].

2. THE MODEL EQUATIONS

The drift turbulence is represented by the following dynamical model:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{v}) = 0 \quad , \tag{1}$$

$$nM\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) + e\,n\,(-\nabla\phi + \underline{v} \times \underline{B}_o) = -\nabla \cdot \underline{\underline{P}} \quad . \tag{2}$$

As we are dealing with low β plasmas, i.e. β smaller than the electron/ion mass ratio, m/M, we consider only electrostatic (longitudinal) oscillations. The confining magnetic field $\underline{B}_o$ is therefore assumed to be constant. Equation (2) for the velocity $\underline{v}$ of ions contains the tensorial pressure $\underline{\underline{P}}$. The effect of finite Larmor radia being important for the ion motion only, may be included in $\underline{\underline{P}}$ if so desired[2,11,12]. The wavelengths involved are much longer than the Debye length, so that we can assume that the density oscillations are quasi-neutral, i.e. $n_i \approx n_e = n$. We furthermore allow for small but non-zero wavenumbers along the magnetic field, i.e. by assuming a large but finite eddy size in this direction. The (warm) electrons can then maintain quasi-neutrality by flowing isothermally along the magnetic field lines, and we can assume that the electrons are Boltzmann distributed, i.e.

$$\kappa\, T_e \,\nabla n = n\, e\, \nabla \phi \tag{3}$$

with $T_e \approx$ const. We assume that the ion motion is isothermal also, i.e. $T_i \approx$ const. Hence, we reduce (1) and (2) to:

$$d_t\psi = -\,\alpha\, \nabla \cdot \underline{v} \tag{4}$$

$$d_t\underline{v} = -\,\alpha\nabla\psi + \frac{e}{M}\, \underline{v} \times \underline{B}_o \tag{5}$$

where $d_t = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$, $\alpha = \sqrt{\kappa(T_e+T_i)/M}$ and $\psi = \alpha \ln (n/n_o)$ with n_o being a normalization constant of dimension density. Note that α and ψ have the dimensions of velocity.

As already mentioned, linear drift waves will have very large wavelength components along the magnetic field lines. The ratio between the perpendicular wavenumber, $k_\perp$, and $k_\parallel$ (along the magnetic field) may be estimated from the relations

$$v_{i,th} < \omega/k_\parallel < v_{e,th}\ , \qquad \omega = k_\perp v_d\ ,$$

or

$$v_{i,th}/v_d < k_\perp/k_\parallel < v_{e,th}/v_d \tag{6}$$

where $v_{i,th}$ and $v_{e,th}$ are the ion and electron thermal velocities, respectively, and v_d is the diamagnetic drift. With a finite ion temperature and a weak density gradient, we can expect strongly field-aligned perturbations as supported by experimental observations[13,14,15]. Under these circumstances we can further simplify our equations (4) and (5) by considering only components perpendicular to the magnetic field.

By introducing vorticity, $\nabla \times \underline{v}$, we can reduce (4) and (5) to a two-dimensional Navier-Stokes equation and find a close similarity between turbulence in magnetized plasma and in incompressible fluids[17]. In this study we shall, however, treat the complete system of equations (4) and (5).

3. CASCADE DECOMPOSITION

We may rewrite our model equations (4) and (5) in the form

$$d_t \psi = - \alpha\, \partial_i\, v_i \tag{7}$$

and

$$d_t\, v_i - \omega_c\, \varepsilon_{ij}\, v_j = -\alpha\, \partial_i \psi \tag{8}$$

in the plane perpendicular to the confining magnetic field; $\omega_c = eB_o/M$ is the ion cyclotron frequency, and $\partial_i = \partial/\partial x_i$, while

$$\{\ \varepsilon_{ij}\ \} = \begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix}\ . \tag{9}$$

Summation over repeated indices is understood throughout. The non-linear system of equations (7)-(8) will be treated by cascade decomposition[6]. As we expect a coupling between various portions of a spectrum, we decompose the velocity $\underline{v}$ into

$$\underline{v}(\underline{x},t) = \bar{\underline{v}}(\underline{x}) + \underline{u}(\underline{x},t) \tag{10}$$

$$\underline{u}(\underline{x},t) = \underline{u}^{(o)}(\underline{x},t) + \underline{u}'(\underline{x},t) \tag{11}$$

where $\bar{\underline{v}}(x)$ is the mean velocity and $\underline{u}^{(o)}$ and $\underline{u}'$ represent macroscopic and random fluctuations, respectively, separated by a wavenumber $\underline{k}$ that serves as an independent variable. $\underline{u}^{(o)}(\underline{k},t)$ and $\underline{u}'(\underline{k},t)$ are components truncated within the large and smaller scales, respectively. The function ψ can be decomposed similarly. This decomposition may in principle be continued with $\underline{u}'$, etc., infinitely, leading to a hierarchy of equations forming the basis for the repeated cascade theory[16,18,19]. In this work, however, we shall limit ourselves to a single decomposition.

The two components in eq. (11) can be screened by an ensemble average

$$< \ldots\ldots >_k \tag{12}$$

scaled at k; this procedure derives the equations for $\underline{u}^{(o)}$, $\underline{u}'$, $\psi^{(o)}$ and ψ' from eqs. (7) and (8) as follows:

$$D_t u_i^{(o)} - \omega_c \varepsilon_{ij} u_j^{(o)} = -u_j^{(o)} \partial_j \bar{v}_i - \alpha \partial_i \psi^{(o)} - \langle u'_j \partial_j u'_i \rangle \tag{13}$$

$$D_t \psi^{(o)} = -u_j^{(o)} \partial_j \bar{\psi} - \alpha \partial_j u_j^{(o)} - \langle u'_j \partial_j \psi' \rangle \tag{14}$$

$$d_t u'_i - \omega_c \varepsilon_{ij} u'_j = -\alpha \partial_i \psi' - u'_s \partial_s (\bar{v}_i + u_i^{(o)}) + \langle u'_j \partial_j u'_i \rangle \tag{15}$$

$$d_t \psi' = -\alpha \partial_j u'_j - u'_s \partial_s (\bar{\psi} + \psi^{(o)}) + \langle u'_j \partial_j \psi' \rangle \tag{16}$$

where $D_t = \partial_t + (\bar{\underline{v}} + \underline{u}^{(o)}) \cdot \nabla$ as distinct from d_t defined earlier. For brevity, we omit the index k on the average sign.

In order to form correlation functions from eqs. (15) and (16) we write

$$u'_i = -(\partial_s(\bar{v}_i+u_i^{(o)}))\int_o^t d\tau U_\omega(t,t-\tau)u'_s(t-\tau)+\frac{e}{M}\int_o^t d\tau U_\omega(t,t-\tau)\mathcal{E}'_i(t-\tau) \tag{17}$$

$$\psi' = -(\partial_s(\bar{\psi}+\psi^{(o)}))\int_o^t d\tau U(t,t-\tau)u'_s(t-\tau)-\alpha\partial_i\int_o^t d\tau U(t,t-\tau)u'_i(t-\tau) \tag{18}$$

where

$$\frac{e}{M}\mathcal{E}_i \equiv -\alpha\partial_i\psi \ , \tag{19}$$

obtaining

$$\begin{aligned}<u'_j\,\partial_j u'_i> = &-<u'_j\,\partial_j(\partial_s(\bar{v}_i+u_i^{(o)}))\int_o^t d\tau U_\omega(t,t-\tau)u'_s(t-\tau)> \\ &+\frac{e}{M}<u'_j\,\partial_j\int_o^t d\tau U_\omega(t,t-\tau)\mathcal{E}'_i(t-\tau)> \ ,\end{aligned} \tag{20}$$

and

$$\begin{aligned}<u'_j\,\partial_j\psi'> = &-<u'_j\,\partial_j(\partial_s(\bar{\psi}+\psi^{(o)}))\int_o^t d\tau U(t,t-\tau)u'_s(t-\tau)> \\ &-\alpha<u'_j\,\partial_j\partial_i\int_o^t d\tau U(t,t-\tau)u'_i(t-\tau)> \ .\end{aligned} \tag{21}$$

The propagators U and U_ω give a Lagrangian representation to the functions following them. Note that U_ω contains the effect of the magnetic field. The macroscopic quantities, $\bar{v}_i + u_i^{(o)}$ and $\bar{\psi} + \psi^{(o)}$ are taken outside the integral sign, because they vary slowly, as compared to the random fluctuations. The last terms in (15) and (16) do not contribute to the correlations (20) and (21) and are therefore omitted in (17) and (18).

Now we assume that a flux of transport is proportional to the gradient of the macroscopic quantity transported, following the transport theory of non-equilibrium thermodynamics. Under these circumstances, we reduce (20) and (21) to

$$<u'_j\, \partial_j u'_i> = -\, \partial_j K'_{js}\, \partial_s(\bar{v}_i + u_i^{(o)}) \tag{22}$$

$$<u'_j\, \partial_j \psi'> = -\, \partial_j \lambda'_{js}\, \partial_s(\bar{\psi} + \psi^{(o)}) \tag{23}$$

by neglecting transports by cross-diffusions. Here

$$K'_{js} = \int_o^t d\tau\, <u'_j\,(t,x)\; U_\omega(t,t-\tau)u'_s(t-\tau)> \tag{24}$$

$$\lambda'_{js} = \int_o^t d\tau\, <u'_j\,(t,x)\; U(t,t-\tau)u'_s\,(t-\tau)> \tag{25}$$

are diffusivities in the presence and absence of the magnetic field, respectively.

In locally homogeneous turbulence, the transport coefficients (24) and (25) can be assumed to be independent of position and simplify (22) and (23) to

$$<u'_j\, \partial_j u'_i> = -\, K'_{js}\, \partial_j \partial_s(\bar{v}_i + u_i^{(o)}) \tag{26}$$

$$<u'_j\, \partial_j \psi'> = -\lambda'_{js}\, \partial_j \partial_s(\bar{\psi} + \psi^{(o)})\ . \tag{27}$$

We should emphasize that the fluxes (26) and (27) are of the gradient type contributing to the transfer across individual spectra of u and ψ fluctuations. However, there are correlations, such as $<u^{(o)}\mathcal{E}^{(o)}>$ and $<u'\mathcal{E}'>$, which play the role of a coupling between the two spectra and will therefore be of non-gradient type. Thus by multiplying eq. (17) by $\frac{e}{M}\,\mathcal{E}'_i$ and averaging we obtain

$$\begin{aligned}\frac{e}{M}<\mathcal{E}'_i\, u'_i> = (\tfrac{e}{M})^2 \int_o^t d\tau\, <\mathcal{E}'_i\,(t,x)\; U_\omega(t,t-\tau)\; \mathcal{E}'_i\,(t-\tau)> \\ -\,\frac{e}{M}\,\partial_s(\bar{v}_i+u_i^{(o)}) \int_o^t d\tau\, <\mathcal{E}'_i\,(t,x)U_\omega(t,t-\tau)u'_s\,(t-\tau)>\ .\end{aligned} \tag{28a}$$

For the reasons mentioned above, we expect that the first term in (28a) is predominant, as it is essential to the coupling between the two spectra, and that the last term is negligible. A similar expression for $\langle\xi_i^{(o)} u_i^{(o)}\rangle$ can be obtained from (28a)

$$\frac{e}{M}\langle\xi_i^{(o)} u_i^{(o)}\rangle = \left(\frac{e}{M}\right)^2 \int_0^t d\tau \langle \xi_i^{(o)}(t,x)\; U_\omega(t,t-\tau)\; \xi_i^{(o)}(t-\tau)\rangle$$
$$\equiv D^{(o)}(t,x) \quad . \tag{28b}$$

The quantities K'_{js} and λ'_{js} are eddy-viscosity and eddy-diffusivity in x-space, i.e. physical space, while $D^{(o)}$ is an eddy-diffusivity in velocity space.

We may reduce U_ω to U, appearing in (24) and (28b), as follows:

$$K'_{js}(t,x) = \int_0^t d\tau \;\langle u'_j(t,x)\; U(t,t-\tau) u'_s(t-\tau)\rangle \cos\omega_c\tau \tag{29}$$

and

$$D^{(o)}(t,x) = \left(\frac{e}{M}\right)^2 \int_0^t d\tau \;\langle\xi_i^{(o)}(t,x)\; U(t,t-\tau)\; \xi_i^{(o)}(t-\tau)\rangle\cos\omega_c\tau \tag{30}$$

Since ω_c is large we only get a significant contribution to the integral from a small region around $\tau \sim 0$, and because $U(t,t) = 1$ we can write (29) and (30) approximately as

$$K'_{js}(t,x) \simeq \frac{\text{const}}{\omega_c}\langle u'_j(t,x)\; u'_s(t,x)\rangle \quad , \tag{31}$$

and

$$D^{(o)}(t,x) \simeq \frac{\text{const.}}{\omega_c}\left(\frac{e}{M}\right)^2 \langle(\xi_i^{(o)}(t,x))^2\rangle \; . \tag{32}$$

A more rigorous treatment would require a determination of U_ω and U by repeated cascade[16-20]. We believe, however, that the approximate expressions (31) and (32) are sufficiently accurate for our purpose.

In order to obtain the equations for energy balance we now multiply eqs. (13) and (14) by $u_i^{(o)}$ and $\psi^{(o)}$, respectively. By making use of eqs. (24)-(27), and after averaging, we obtain

$$\langle u_i^{(o)} D_t u_i^{(o)} \rangle = K_{js}^{(o)} \partial_s \bar{v}_i \partial_j \bar{v}_i + D^{(o)} + K'_{js} \langle u_i^{(o)} \partial_s \partial_j u_i^{(o)} \rangle \, , \tag{33}$$

and

$$\langle \psi^{(o)} D_t \psi^{(o)} \rangle = \lambda_{js}^{(o)} \partial_s \bar{\psi} \partial_j \bar{\psi} - \alpha \langle \psi^{(o)} \partial_j u_j^{(o)} \rangle + \lambda'_{js} \langle \psi^{(o)} \partial_s \partial_j \psi^{(o)} \rangle \, . \tag{34}$$

We have applied expressions for $\langle u_j^{(o)} u_i^{(o)} \rangle$ and $\langle u_j^{(o)} \psi^{(o)} \rangle$ that are obtainable in the same way as (26) and (27) using eqs. (13) and (14) as in eqs. (20) and (21). This operation involves $K_{js}^{(o)}(t,x)$ and $\lambda_{js}^{(o)}(t,x)$ in a form similar to (24) and (25) but with a zeroth rank. Making use of the identities

$$\langle u_i^{(o)} \partial_s \partial_j u_i^{(o)} \rangle = \frac{1}{2} \partial_s \partial_j \langle (u_i^{(o)})^2 \rangle - \langle \partial_s u_i^{(o)} \partial_j u_i^{(o)} \rangle \tag{35}$$

$$\langle \psi^{(o)} \partial_s \partial_j \psi^{(o)} \rangle = \frac{1}{2} \partial_s \partial_j \langle (\psi^{(o)})^2 \rangle - \langle \partial_s \psi^{(o)} \partial_j \psi^{(o)} \rangle \tag{36}$$

and

$$\begin{aligned} \alpha \langle \psi^{(o)} \partial_j u_j^{(o)} \rangle &= \alpha \partial_j \langle u_j^{(o)} \psi^{(o)} \rangle - \alpha \langle u_j^{(o)} \partial_j \psi^{(o)} \rangle \\ &= \alpha \partial_j \langle u_j^{(o)} \psi^{(o)} \rangle + D^{(o)} \end{aligned} \tag{37}$$

we rewrite eqs. (33) and (34) as

$$\begin{aligned} \langle u_i^{(o)} D_t u_i^{(o)} \rangle = {} & K_{js}^{(o)} \partial_s \bar{v}_i \partial_j \bar{v}_i + D^{(o)} + K'_{js} \frac{1}{2} \partial_s \partial_j \langle (u_i^{(o)})^2 \rangle \\ & - K'_{js} \langle \partial_s u_i^{(o)} \partial_j u_i^{(o)} \rangle \end{aligned} \tag{38}$$

$$\begin{aligned} \langle \psi^{(o)} D_t \psi^{(o)} \rangle = {} & \lambda_{js}^{(o)} \partial_s \bar{\psi} \partial_j \bar{\psi} - D^{(o)} - \alpha \partial_j \langle u_j^{(o)} \psi^{(o)} \rangle \\ & + \lambda'_{js} \frac{1}{2} \partial_s \partial_j \langle (\psi^{(o)})^2 \rangle - \lambda'_{js} \langle \partial_s \psi^{(o)} \partial_j \psi^{(o)} \rangle \, . \end{aligned} \tag{39}$$

The first term in (38) and (39) is a production function, the second represents a coupling, while the last term accounts for the transfer across the spectrum. The remaining terms arise from non-equilibrium and inhomogeneity. In the universal range of the spectrum, which we will consider in the following, these terms will not play any role. They are important, however, in the non-universal range. The gradients of $\nabla\bar{v}$ and $\nabla\bar{\psi}$ represents inhomogeneities.

In the following, we assume that the traces in the diffusivity tensors are more important than the off-diagonal components, reducing the production functions to

$$P_u^{(o)} \equiv K_{js}^{(o)} \partial_s \bar{v}_i \partial_j \bar{v}_i \cong K^{(o)} \Gamma_u^2 \tag{40a}$$

$$P_\psi^{(o)} \equiv \lambda_{js}^{(o)} \partial_s \bar{\psi} \partial_j \bar{\psi} \cong \lambda^{(o)} \Gamma_\psi^2 \tag{40b}$$

and the transfer functions to

$$T_u^{(o)} \equiv K'_{js} \langle \partial_s u_i^{(o)} \partial_j u_i^{(o)} \rangle \cong K' R^{(o)} \tag{41a}$$

$$T_\psi^{(o)} \equiv \lambda'_{js} \langle \partial_s \psi^{(o)} \partial_j \psi^{(o)} \rangle \cong \lambda' J^{(o)} \tag{41b}$$

with

$$\Gamma_u^2 \equiv (\partial_j \bar{v}_i)^2 \tag{42a}$$

$$\Gamma_\psi^2 \equiv (\partial_j \bar{\psi})^2 \tag{42b}$$

and

$$R^{(o)} \equiv \langle (\partial_j u_i^{(o)})^2 \rangle \tag{42c}$$

$$J^{(o)} \equiv \langle (\partial_j \psi^{(o)})^2 \rangle \quad . \tag{42d}$$

With these definitions, the energy equations (38) and (39) become

$$\langle u_i^{(o)} D_t u_i^{(o)} \rangle = P_u^{(o)} + D^{(o)} - T_u^{(o)} \tag{43}$$

$$\langle \psi^{(o)} D_t \psi^{(o)} \rangle = P_\psi^{(o)} - D^{(o)} - T_\psi^{(o)} \quad . \tag{44}$$

These equations form the framework for our spectral analysis.

4. SPECTRAL STRUCTURE

4.1. Classification of Spectral Subranges

We shall distinguish between universal and non-universal ranges in a spectrum and shall be concerned with the universal range exclusively. We can subdivide the universal range into production, coupling, and inertia subranges. The individual subranges are investigated separately.

4.1.1. The Production Subrange

Mean gradients of velocity and potential will feed energy into the fluctuations for further transfer across the individual spectra. In order to describe this transport explicitly we write the energy equations in the following differential form

$$\Gamma_u^2 \; \dot{K}^{(o)} - K' \; \dot{R}^{(o)} - \dot{K}' \; R^{(o)} = 0 \tag{45}$$

$$\Gamma_\psi^2 \; \dot{\lambda}^{(o)} - \lambda' \; \dot{J}^{(o)} - \dot{\lambda}' \; J^{(o)} = 0 \tag{46}$$

obtained by retaining the production and transfer functions in (43) and (44). The upper dot represents a differentiation with respect to k. Introducing the spectral distribution F(k) and G(k) of u and ψ, such that

$$\frac{1}{2} \langle (u_i^{(o)})^2 \rangle = \int_0^k dk' \; F(k') \tag{47}$$

$$\frac{1}{2} \langle (\psi^{(o)})^2 \rangle = \int_0^k dk' \; G(k') \tag{48}$$

we have

$$R^{(o)} \equiv \langle (\partial_j u_i^{(o)})^2 \rangle = 2 \int_0^k dk' \; k'^2 \; F(k') \tag{49}$$

$$J^{(o)} \equiv \langle (\partial_j \psi^{(o)})^2 \rangle = 2 \int_0^k dk' \; k'^2 \; G(k') \; . \tag{50}$$

As the production occurs at low wavenumbers we may neglect the last terms in (45) and (46) on account of small $R^{(o)}$ and $J^{(o)}$.

The simplified equations then become

$$\Gamma_u^2 \dot{K}^{(o)} = K' \dot{R}^{(o)} , \tag{51}$$

$$\Gamma_\psi^2 \dot{\lambda}^{(o)} = \lambda' \dot{J}^{(o)} , \tag{52}$$

yielding the solutions

$$F(k) = \text{const.} \Gamma_u^2 k^{-3} , \tag{53}$$

$$G(k) = \text{const.} \Gamma_\psi^2 k^{-3} . \tag{54}$$

We shall not attempt to determine the numerical constants.

4.1.2. The Coupling Subrange

The governing terms in this subrange are the coupling and the transfer functions. From eqs. (43) and (44) we have

$$D^{(o)} - T_u^{(o)} = - \varepsilon_u \tag{55}$$

$$-D^{(o)} - T_\psi^{(o)} = - \varepsilon_\psi \tag{56}$$

or with the substitution of (19), (31) and (32),

$$\lambda_c J^{(o)} - K' R^{(o)} = - \varepsilon_u \tag{57}$$

$$-\lambda_c J^{(o)} - \lambda' J^{(o)} = - \varepsilon_\psi \tag{58}$$

where ε_u and ε_ψ are the rates of energy dissipation from u and ψ fluctuations, respectively, and

$$\lambda_c = \text{const.} \alpha^2/\omega_c \tag{59}$$

is found to be the Bohm diffusion[20]. In order to describe the exchange processes more explicitly, we again rewrite (57) and (58) in differential form

$$\lambda_c \dot{J}^{(o)} - \dot{K}' R^{(o)} - K' \dot{R}^{(o)} = 0 \quad , \tag{60}$$

$$-\lambda_c \dot{J}^{(o)} - \dot{\lambda}' J^{(o)} - \lambda' \dot{J}^{(o)} = 0 \ . \tag{61}$$

Because the ϕ spectrum is dissipated at a rate controlled by the Bohm diffusion, we can introduce the following approximations

$$R^{(o)} = 0, \quad J^{(o)} = J \quad \text{and} \quad \lambda' << \lambda_c \ .$$

These approximations reduce (60) and (61) to

$$\lambda_c \dot{J}^{(o)} - K' \dot{R}^{(o)} = 0 \tag{62}$$

$$-\lambda_c \dot{J}^{(o)} - \dot{\lambda}' J = 0 \tag{63}$$

or, after addition,

$$-K' \dot{R}^{(o)} - \dot{\lambda}' J = 0 \ . \tag{64}$$

Substitution of (31) into (64) gives

$$- \frac{\langle u'^2 \rangle \dot{R}^{(o)}}{\dot{\lambda}'} = \omega_o^3 \quad , \tag{65}$$

where

$$\omega_o^3 \equiv \omega_c J \ . \tag{66}$$

As the left-hand side of eq. (65) is a function of F and k alone, we obtain the solution

$$F(k) = \text{const.}\, \omega_o^3 \, k^{-3} \ . \tag{67}$$

On the other hand, eq. (62) gives, by the definition (31),

$$2 k^2 G(k) = \text{const.} \frac{\langle u'^2 \rangle}{\lambda_c \omega_c} \dot{R}^{(o)} \quad , \tag{68}$$

On substituting the solution (67) for F(k), we find, from eq. (68),

$$G(k) = \text{const.}\,(\omega_o^4/\lambda_c\omega_c)k^{-5} \tag{69}$$

corresponding to a spectrum for the electrostatic field

$$G_{\mathcal{E}}(k) = \text{const.}\,(\alpha^2\omega_o^4/\lambda_c\omega_c)k^{-3} \tag{70}$$

which is defined so that

$$\frac{1}{2}\,(e/M)^2 \,\langle\, \mathcal{E}_i^{(o)\,2} \,\rangle = \int_o^k dk'\,G_{\mathcal{E}}(k')\ . \tag{71}$$

We can conclude that the ψ-spectrum (69) falls off much faster than the u-spectrum (67) in the present subrange, because the Bohm diffusion is effective in the ψ-spectrum only.

4.1.3. Inertia Subrange

The inertia subrange is characterized by a constant transfer of energy across the spectrum, i.e. from eqs. (55) and (56)

$$T_u^{(o)} = \varepsilon_u \tag{72}$$

$$T_\psi^{(o)} = \varepsilon_\psi \tag{73}$$

or, with the definition (41),

$$K'\ R^{(o)} = \varepsilon_u, \tag{74}$$

$$\lambda'\ J^{(o)} = \varepsilon_\psi. \tag{75}$$

In terms of the spectral function, (31) takes the form

$$K' = \frac{\text{const.}}{\omega_c}\int_k^\infty dk'\ F(k') \tag{76}$$

which combined with (49) permits us to rewrite (74) as

$$\frac{\text{const.}}{\omega_c}\int_k^\infty dk'\,F(k')\ 2\int_o^k dk''k''^2\,F(k'') = \varepsilon_u\ , \tag{77}$$

or

$$\int_0^k dk'k'^2F(k') = \frac{\varepsilon_u \omega_c}{2\ \text{const.}} \frac{1}{\int_k^\infty dk'F(k')} . \tag{78}$$

After differentiation with respect to k, we obtain

$$k^2F(k) = \frac{\varepsilon_u \omega_c}{2\ \text{const.}} \frac{F(k)}{\left[\int_k^\infty dk'F(k')\right]^2} , \tag{79}$$

or

$$\int_k^\infty dk'F(k') = (\varepsilon_u \omega_c/2\ \text{const.})^{1/2}\ k^{-1} . \tag{80}$$

Differentiation of (80) finally yields

$$F(k) = (\varepsilon_u \omega_c/2\ \text{const.})^{1/2}\ k^{-2}. \tag{81}$$

On substituting (81) into (75), we obtain

$$G(k) = \text{const.} \frac{\varepsilon_\psi}{(\varepsilon_u \omega_c)^{1/4}}\ k^{-3/2} . \tag{82}$$

As the ψ spectrum is quickly dissipated by Bohm diffusion, as mentioned in section 4.1.2, we may expect under certain circumstances that the inertia subrange (82) may not be fully developed.

5. CONCLUSION

For comparison, we collected data on turbulent spectra reported in the literature[13,21-28]. In most experiments the spectra are measured as a function of frequency rather than wavenumber, thus introducing the problem of Eularian-Lagrangian transformation. This can be solved by a generalization of the expressions (29) and (30). In this connection, if the streaming velocity included in the propagator U predominates, we obtain a linear relation between the spectral functions in ω and k spaces.

This amounts to Taylor's hypothesis, or frozen turbulence, which is very widely accepted in the interpretation of turbulence measurements in plasmas and fluids[29].

It will be noted that several of the measured spectra refer to density fluctuations, while our model equations (4) and (5) indicate that velocity and potential fluctuations are more easily tractable quantities for theoretical analysis. Under most experimental conditions[25,30] the measured spectra of density and potential have quite similar structure.

Figure 1 shows spectra obtained in conventional Q-devices reproduced from refs. 23-25. The reported spectra show amplitude variation as a function of frequency, so the spectral index should be multiplied by two before comparison with the theoretical predictions. We emphasize the measurements of refs. 24 and 25 where a large $\underline{E}x\underline{B}$ rotation of the entire plasma column renders Taylor's hypothesis effective.

Figure 2 shows turbulent spectra for the density fluctuations obtained in a hot-cathode reflex arc (ref. 13) and the Etude stellarator (ref. 21). Spectral measurements of the fluctuating electric field in the Zeta discharge (ref. 22) are shown in fig.3.

Finally, fig. 4 shows a spectral analysis of turbulent density fluctuations in barium plasmas released in the upper atmosphere (ref. 28). Because the data were reduced by some photographic technique it was possible, in this case, to present the spectrum as a function of wavenumber.

In conclusion, we summarize the comparison between theory and experimental results as follows:

a) Production subrange. The predicted spectral index $\alpha = 3$ for potential fluctuations is well demonstrated on figs. 1 and 4. On fig. 2 this index is not so clearly presented, but also here the spectra are consistently flatter (rather than steeper) than in the coupling subrange. We note that the measurements reported in ref. 25 fit well with the expected spectrum for this subrange even though the B-field was rather inhomogeneous in this experiment.

b) Coupling subrange. We find the spectral shape of the density fluctuations shown on figs. 1-4 to be in excellent agreement with theoretical predictions. (See also ref. 13). The measurements reported in refs. 26 and 27 (not shown on figures) add further weight to our results. In particular, we note that the power spectrum of the fluctuating electric field in Zeta (see fig. 3) agrees with the theoretical predictions for this subrange.

c) Inertia subrange. Unfortunately, we have only been able to find one measurement that also covers this subrange (see fig. 1 and ref. 24). Even this measurement shows only a portion of the spectrum. We find good agreement with theoretical predictions for this single case, but we should like to emphasize the comment made in subsection 4.1.3; in certain circumstances the inertia subrange of the potential fluctuations need not be fully developed.

We thus feel, on the basis of the arguments and measurements presented here, that our theoretical predictions for the spectral shape of the potential fluctuations are in good agreement with experimental results. We find it particularly comforting that our data refer to widely differing experimental conditions and not to one particular device. To the best of our knowledge, experimental results for the velocity fluctuations are not yet available. An investigation of our predicted parameter dependence of the turbulent fluctuations will also have to await future experiments.

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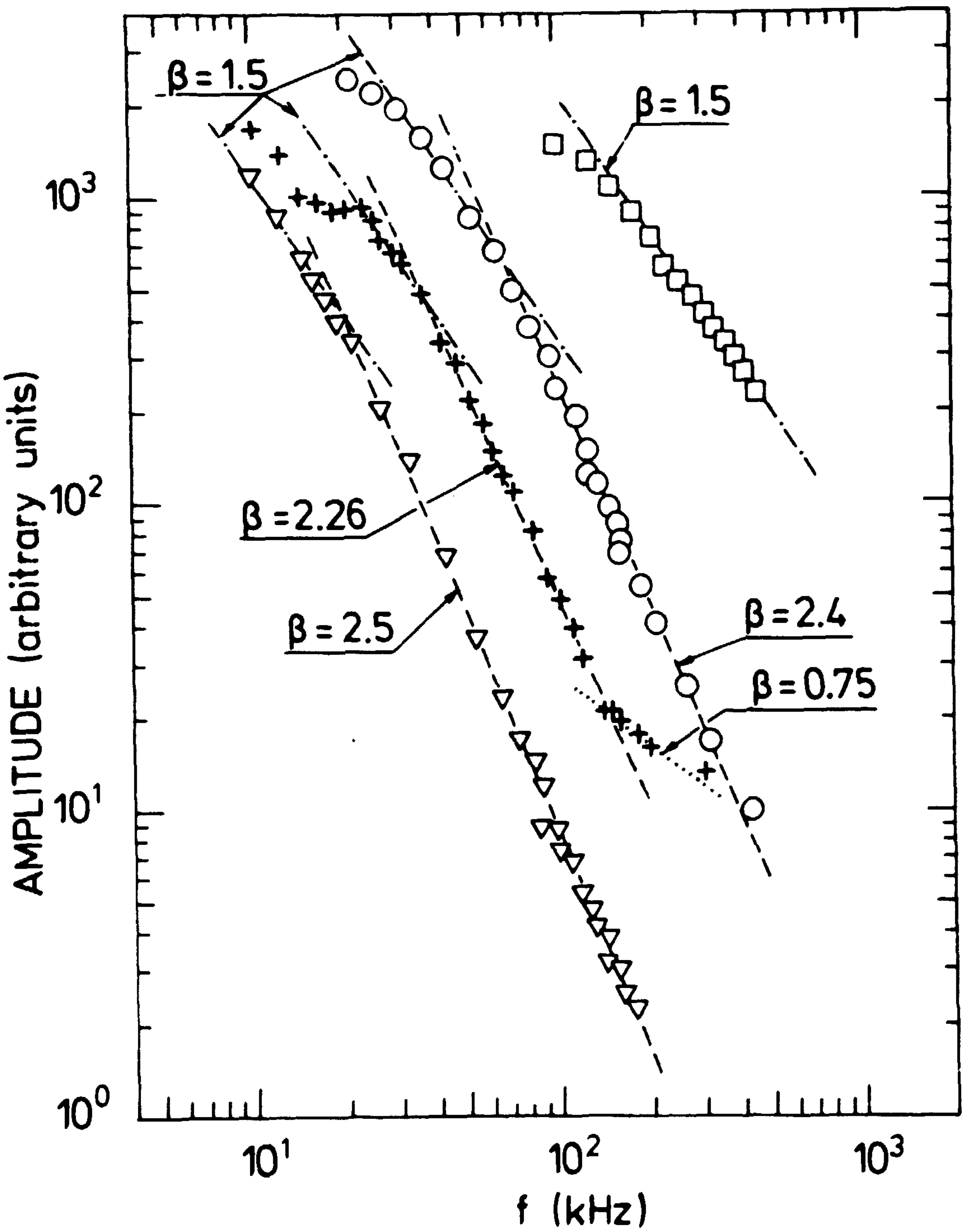

Fig. 1. Turbulent spectra obtained using Q-devices. All curves show the amplitude in the fluctuations as a function of frequency. The curves are reproduced from refs. 23 (∇ and o), 24 (+) and 25 (□). The measurements of refs. 23 and 25 refer to fluctuations in floating potential, while in the experiment reported in ref. 24 the ion saturation current to a Langmuir probe is measured. The spectral index for the amplitude variations is denoted by β, i.e $A(f) \sim f^{-\beta}$.

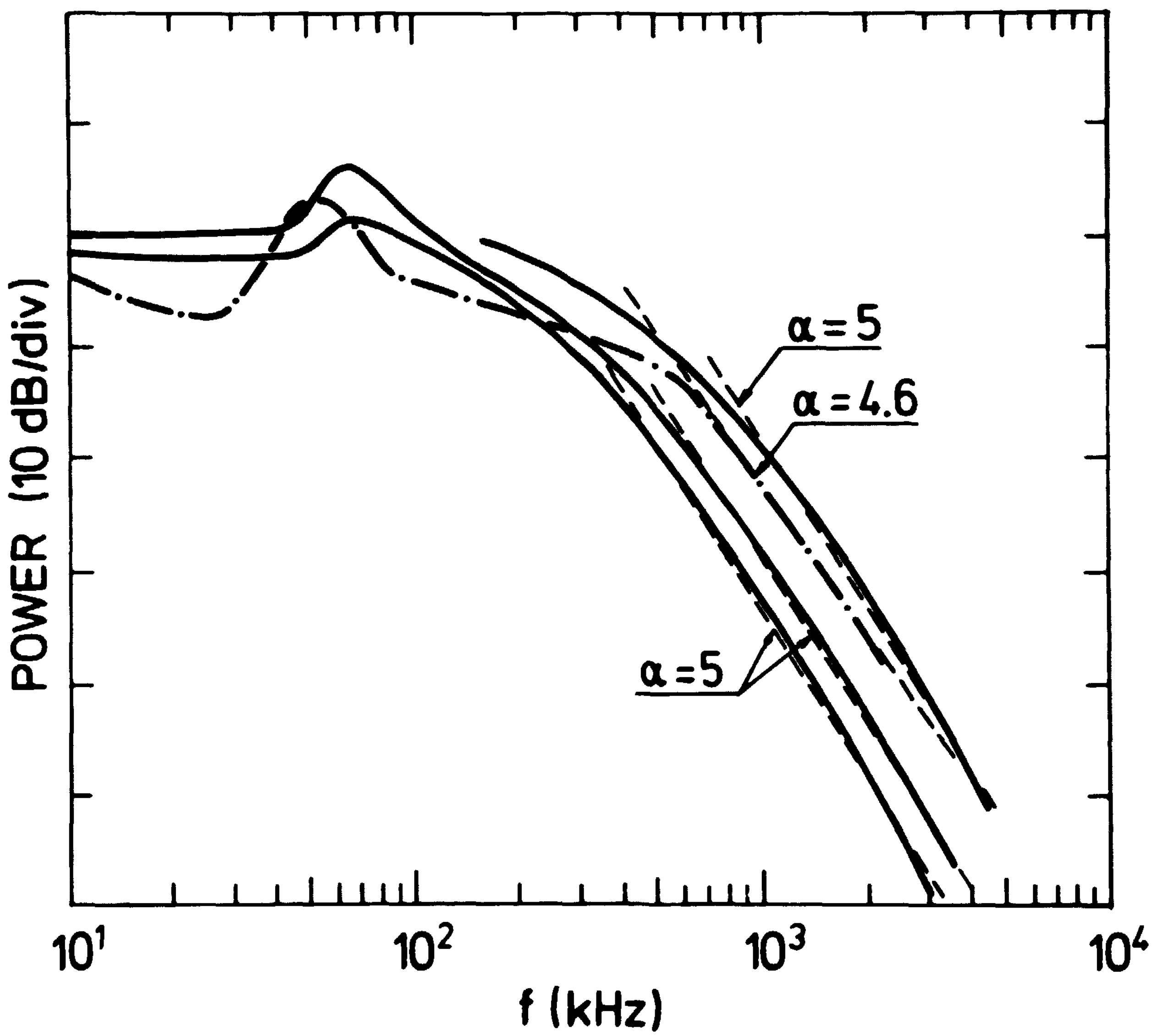

Fig. 2. Turbulent power spectra obtained in a hot-cathode reflex arc[13] (-·-·-) and in the Etude stellerator[21] (——). The measurements refer to density fluctuations. The spectral index for the power spectrum is denoted by α, i.e. $P(f) \sim f^{-\alpha}$.

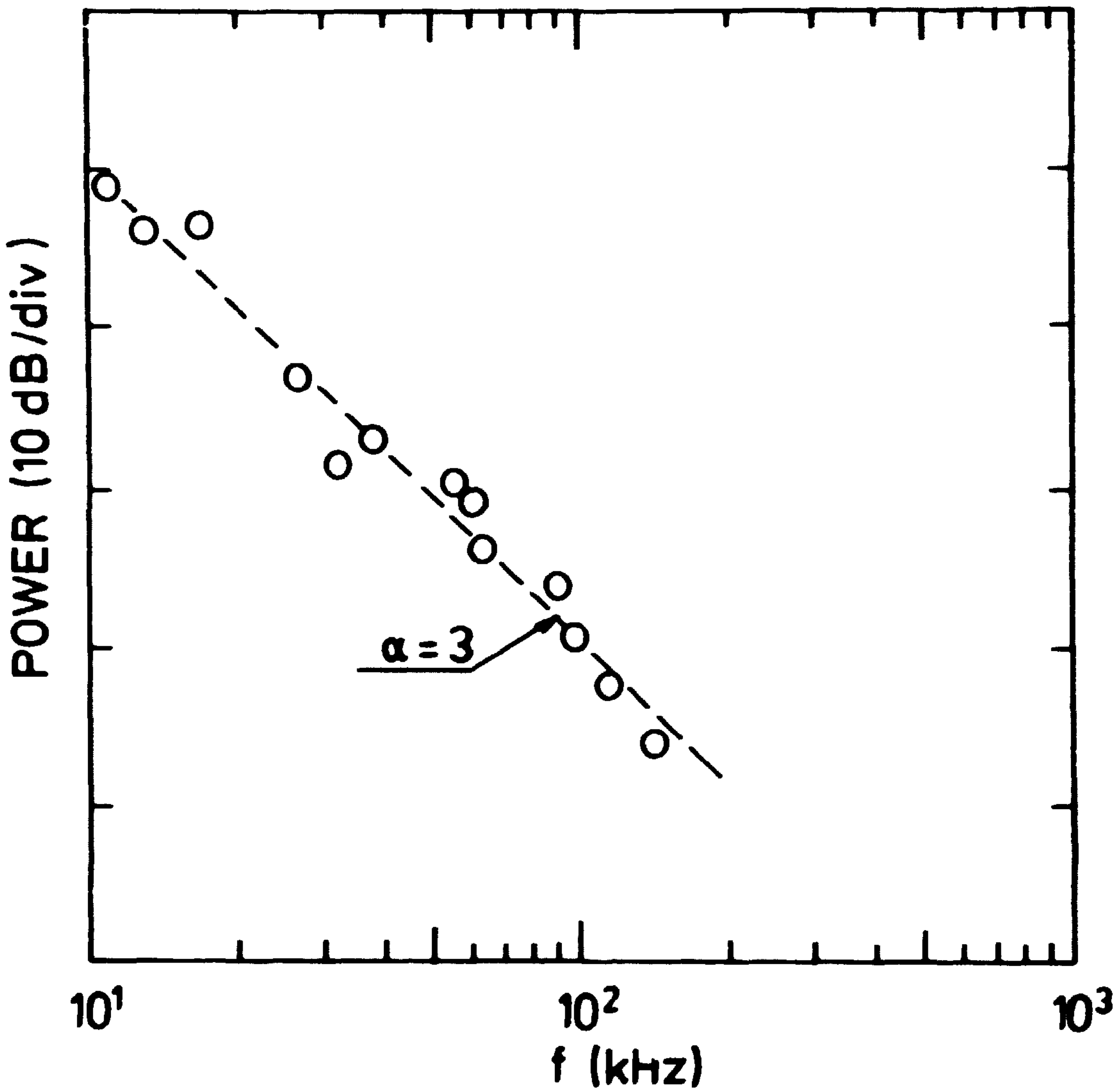

Fig. 3. Power spectrum for the fluctuating electric field measured in the Zeta discharge[22].

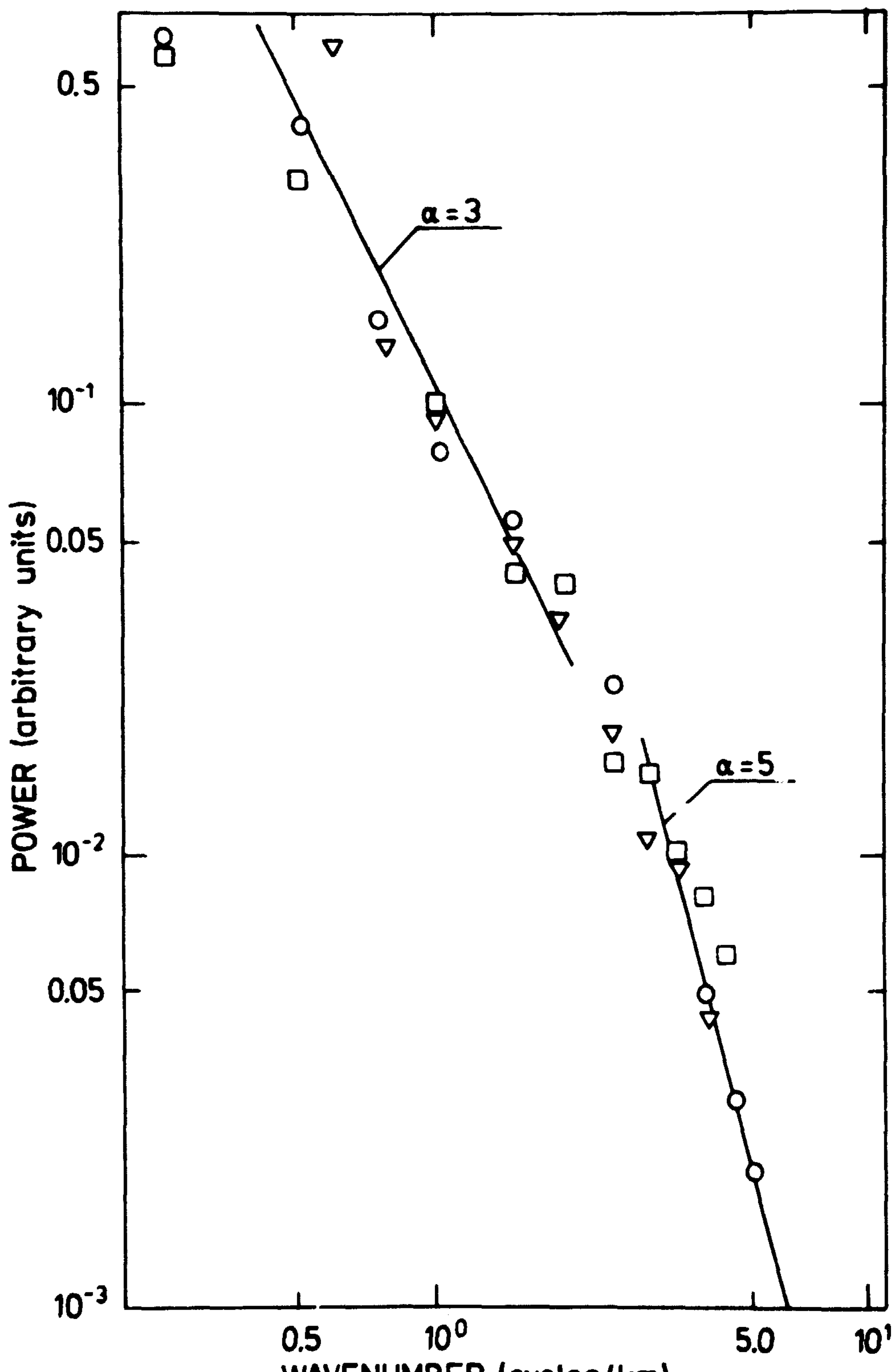

Fig. 4. Turbulent density fluctuations in a barium plasma artificially released in the upper atmosphere[28]. The spectrum is shown as a function of wavenumber.

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