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A Non-linear Kinematic Hardening Function.

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Abstract

Based on the classical theory of plasticity and accepting the von Mises criterion as the initial yield criterion, a non-linear kinematic hardening function applicable both to Melan-Prager's and to Ziegler's hardening rule is proposed. This non-linear hardening function is determined by means of the uniaxial stress-strain curve, and any such curve is applicable. The proposed hardening function considers both the problem of general reversed loading, and a smooth change in the behaviour from one plastic state to another nearlying plastic state is obtained. A review of both the kinematic hardening theory and the corresponding non-linear hardening assumptions is given, and it is shown that material behaviour is identical whether Melan-Prager's or Ziegler's hardening rule is applied, provided that the von Mises yield criterion is adopted.

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INTRODUCTION

Accurate calculation of non-proportional inelastic behaviour, including cycling of metals under multiaxial stress states, is of importance in many structures, and notably in aircraft and nuclear applications. It is the purpose of this report to propose, within the classical theory of plasticity, a new non-linear kinematic hardening function, which considers the problem of general reversed loading, and where a smooth change in the behaviour from one plastic state to another nearlying plastic state is obtained too. In addition, it is shown that material behaviour is identical whether Melan-Prager's or Ziegler's hardening rule is applied, provided that the von Mises yield criterion is adopted. As no recent review of both the kinematic hardening theory and the corresponding non-linear hardening assumptions seems to exist, such a review is included in the following, so that the proposed non-linear hardening function can be evaluated on a suitable background.

KINEMATIC HARDENING

It is commonly known that for loadings that are far from proportional, isotropic hardening is insufficient and kinematic hardening, where the loading surfaces translate as rigid surfaces maintaining their orientation in the stress space, provides an approximation to reality that seems more promising. In particular, kinematic hardening provides a method of considering the Bauschinger effect observed in most metal behaviours. If the yield surface for an initially isotropic material is described by

$$f(\sigma_{ij}) = \kappa \tag{1}$$

where σ_{ij} is the stress tensor, and κ is a constant, then, assuming kinematic hardening, the loading surfaces are given by

$$f(\sigma_{ij}-\alpha_{ij}) = \kappa \tag{2}$$

where f is the same function as in eq. (1), and where the symmetric tensor α_{ij} describes the total translation of the centre of the loading surface in the stress space. Fig. 1 illustrates the change of the loading surface from position 1 to position 2 due to hardening. 0 is the origin of the stress space, and C is the centre of the loading surface 1, which shifts to C' during the hardening. P denotes the actual stress point located on loading

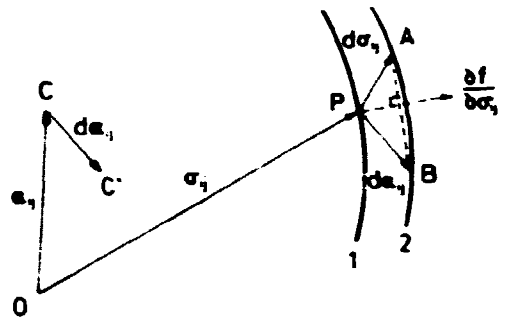

Fig. 1.

surface 1, and as point A is located on loading surface 2 with the centre C', point B is also located here due to eq.(2).

If we accept the normality condition by, e.g., using Drucker's postulate for stable material behaviour [1], then

$$d\varepsilon_{ij}^{P} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{3}$$

where $d\varepsilon_{ij}^{P}$ denotes the differential of the plastic strain tensor, and $d\lambda$ is a positive scalar function during loading. Thus, projecting $d\sigma_{ij}$ on the outer normal at point P given by $\partial f/\partial\sigma_{ij}$ and using eq.(3), we obtain

$$(d\sigma_{ij} - c\,d\varepsilon_{ij}^{P}) \frac{\partial f}{\partial \sigma_{ij}} = 0 \tag{4}$$

where c is a positive scalar function depending in general on the loading history and the present loading. It should be emphasized that eq.(4) implies no further assumptions than those connected with eqs.(2) and (3). By menas of eq.(4) we find

$$d\varepsilon_{ij}^{P} \frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{c} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \tag{5}$$

and elimination of $d\varepsilon_{ij}^{P}$ by means of eq.(3) implies

$$d\lambda = \frac{1}{c} \frac{\dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\dfrac{\partial f}{\partial \sigma_{kl}} \dfrac{\partial f}{\partial \sigma_{kl}}} \tag{6}$$

i.e. $d\lambda$ is determined by the hardening function c, once the loading function is known. It now remains to complete the equations required by determining the tensor $d\alpha_{ij}$.

Using eq.(2), the consistency equation states that

$$(d\sigma_{ij}-d\alpha_{ij})\frac{\partial f}{\partial \sigma_{ij}} = 0 \tag{7}$$

i.e. line AB in fig. 1 is orthogonal to the normal at point P, given by $\partial f/\partial\sigma_{ij}$. Thus, eq.(7) determines the projection of $d\alpha_{ij}$ on the normal at point P, and $d\alpha_{ij}$ is then completely known, once the direction of $d\alpha_{ij}$ is chosen. The concept of kinematic hardening is traditionally attributed to Prager [2], [3], who assumed that the instantaneous translation of the loading surface was orthogonal to the surface at the stress point, which means that $d\alpha_{ij}$ is proportional to $d\varepsilon_{ij}{}^{P}$. Use of eqs.(4) and (7) then gives Prager's hardening rule:

$$d\alpha_{ij} = c d\varepsilon_{ij}{}^{P} \tag{8}$$

where c in Prager's concept was considered a constant, i.e. from eq.(8) it follows that

$$\alpha_{ij} = c\varepsilon_{ij}{}^{P} \tag{9}$$

It is interesting to note that the theory of kinematic hardening was in fact formulated in a much earlier work by Melan [4]. While Melan stated the theory in precise mathematical terms by proposing eqs.(2), (7) and (8) (also considering c as a constant), whereby eqs.(4) and (6) were obtained using eq.(3), Prager's formulation [2], [3] was given in more qualitative terms. It seems therefore reasonable to call eq.(8) Melan-Prager's hardening rule.

Budansky [5] noted apparent inconsistencies in the use of eq.(8) when applied to the state of plane stress, and Hodge [6], [7] showed that these inconsistencies appear when eq.(8) is applied directly to the state of plane stress. Perrone and Hodge [8], [9] pointed out that eq.(8) should always be applied in the full 9-dimensional stress space and termed this complete kinematic hardening in contrast to the direct kinematic hardening, where the translation of the loading surface is orthogonal to the loading surface at the stress point in the actual subspace of the full stress space. Perrone and Hodge [9] compared direct and complete kinematic hardening when applied to plate problems, and even though they may give almost quantitatively similar solutions, the direct hardening rule implies certain inconsistencies as, e.g., the plastic incompressibility for a von Mises material is not obtained. In a detailed investigation Shield and Ziegler [10] found that if Melan-Prager's hardening rule is applied in the full 9-dimensional stress space, then the loading surfaces in subspaces of this stress space may change their shape during loading, and the translation of the loading surfaces may not be in the direction of the outer normal in the actual subspace. These two circumstances, both contrary to the assumptions of direct hardening, can be illustrated by considering a subspace, denoting the non-zero stresses corresponding to that subspace by $\sigma_{ij,1}$ and the remaining zero stresses of the stress tensor by $\sigma_{ij,2}$. Thus, the yield surface is described by

$$f(\sigma_{ij,1}, \sigma_{ij,2} = 0) = h(\sigma_{ij,1}) = \kappa \tag{10}$$

and $d\alpha_{ij}$ is determined by

$$d\alpha_{ij,1} = c d\varepsilon_{ij,1}^{p} = c\, d\lambda \frac{\partial f}{\partial \sigma_{ij,1}} = c\, d\lambda \frac{\partial h}{\partial \sigma_{ij,1}}$$

$$d\alpha_{ij,2} = c d\varepsilon_{ij,2}^{p} = c\, d\lambda \frac{\partial f}{\partial \sigma_{ij,2}}$$

with obvious notation. Even though $\sigma_{ij,2}$ is equal to zero, $d\alpha_{ij,2}$ will in general be non-zero, and the loading surfaces will therefore be described by

$$f(\sigma_{ij,1}-\alpha_{ij,1}, -\alpha_{ij,2}) = \kappa$$

and this equation cannot in general be expressed by the function h defined by eq.(10), i.e. the loading surface may change its shape in the actual subspace during hardening. Besides, even if no change of shape occurs, the translation of the loading surface may occur in a direction different from the outward normal in the subspace determined by $d\varepsilon_{ij,1}^{p}$, as $d\varepsilon_{ij,2}^{p}$ implies a translation in the subspace, which in general is non-proportional to $d\varepsilon_{ij,1}^{p}$.

Thus, Melan-Prager's hardening rule is not invariant with respect to reductions in dimensions. Even though this is physically acceptable, it is mathematically inconvenient, and therefore Ziegler [11] proposed another hardening rule, which is invariant with respect to reductions in dimensions, and which subsitutes eq.(8) by

$$d\alpha_{ij} = (\sigma_{ij}-\alpha_{ij})\, d\mu \tag{11}$$

where the scalar function $d\mu$ is positive. Geometrically, eq.(11) means that the translation of the loading surface occurs in the direction of the vector CP connecting the centre C of the loading surface with the actual stress point P, fig. 1. Combining eqs.(7) and (11) gives

$$d\mu = \frac{\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{(\sigma_{kl}-\alpha_{kl}) \frac{\partial f}{\partial \sigma_{kl}}}$$

Using the earlier notation, it is easily shown that Ziegler's hardening rule, eq.(11), is invariant with respect to reductions in dimensions as $d\alpha_{ij,2}$, corresponding to the zero stresses $\sigma_{ij,2}$, is also zero. Clavout and Ziegler [12] made a comparison in various subspaces between Melan-Prager's and Ziegler's hardening rule; they stated that even though the rules do not in general coincide, there will not be much difference numerically. A general discussion of eqs.(8) and (11) is also found in a work by Naghdi [13].

To determine $d\lambda$ in eq.(3), Ziegler [11] assumed that eq.(4) applies just as in the case of Melan-Prager's hardening rule, where eq.(4) follows from eqs.(7) and (8). However, this is an unnecessary assumption as eq.(4) applies in general as earlier mentioned, i.e. also for Ziegler's hardening rule $d\lambda$ is determined by eq.(6).

In the following, we will restrict ourselves to the use of von Mises criterion, which represents the initial yielding with sufficient accuracy, and which, due to its lack of singularities, is mathematically attractive. Then eq.(1) takes the form

$$\left(\frac{3}{2} s_{ij} s_{ij}\right)^{\frac{1}{2}} = \sigma_o$$

where the deviatoric stress tensor s_{ij} is defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$$

and σ_o is the yield stress for uniaxial tensile loading. Eq.(2) becomes

$$f(\sigma_{ij}-\alpha_{ij}) = \left(\frac{3}{2} s_{ij}' s_{ij}'\right)^{\frac{1}{2}} = \sigma_o \tag{12}$$

where the deviatoric translation tensor α_{ij}^D, the reduced stress tensor σ_{ij}' and the reduced deviatoric stress tensor s_{ij}' are defined by

$$\alpha_{ij}^D = \alpha_{ij} - \frac{1}{3} \delta_{ij} \alpha_{kk}$$

$$\sigma_{ij}' = \sigma_{ij} - \alpha_{ij}$$

$$s_{ij}' = \sigma_{ij}' - \frac{1}{3} \delta_{ij} \sigma_{kk}' = s_{ij} - \alpha_{ij}^D$$

From eq.(12) we obtain

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3\, s_{ij}'}{2\, \sigma_o} \tag{13}$$

By means of eq.(7), eq.(4) is equivalent to

$$(d\alpha_{ij} - c\, d\varepsilon_{ij}{}^{p}) \frac{\partial f}{\partial \sigma_{ij}} = 0$$

and use of eqs.(3), (11), (13) and (12) in the above equation implies that

$$d\mu = \frac{3}{2\sigma_o} c\, d\lambda \tag{14}$$

Using eqs. (12) and (13) in eq.(6) we find

$$d\lambda = \frac{1}{c\sigma_o} s_{ij}'\, d\sigma_{ij}$$

which by means of eq.(3) implies that

$$d\varepsilon_{ij}{}^{p} = \frac{3}{2c\sigma_o{}^2} s_{kl}'\, s_{ij}'\, d\sigma_{kl} \tag{15}$$

From eq.(14) we obtain

$$d\mu = \frac{3}{2\sigma_o{}^2} s_{ij}'\, d\sigma_{ij} \tag{16}$$

In the following, the indices MP and Z refer to Melan-Prager's and Ziegler's hardening rule, respectively. Combination of eqs.(8) and (15) yields

$$d\alpha_{ij,MP} = \frac{3}{2\sigma_o{}^2}\, s_{kl}'\, s_{ij}'\, d\sigma_{kl} \tag{17}$$

Eqs.(11), (16) and (17) imply

$$d\alpha_{ij,Z} = d\alpha_{ij,MP} + \frac{1}{2\sigma_o{}^2}\, \delta_{ij}\, s_{kl}'\, d\sigma_{kl} \tag{18}$$

Hereby we obtain

$$d\alpha^{D}_{ij,Z} = d\alpha^{D}_{ij,MP} = d\alpha_{ij,MP} \tag{19}$$

which means that for a given stress history, we have

$$s'_{ij,MP} = s'_{ij,Z}$$

Eq. (15) then leads to

$$d\varepsilon_{ij,MP}^{\ P} = d\varepsilon_{ij,Z}^{\ P}$$

It has been assumed above that the hardening function c is a function of $\alpha_{ij}^{\ D}$, $\varepsilon_{ij}^{\ P}$, σ_{ij} and $d\sigma_{ij}$.

We have hereby obtained the important result that, for a given stress history, the material response is identical whether Melan-Prager's or Ziegler's hardening rule is applied, provided that the von Mises yield criterion is adopted. Only a translation of the loading surface along the hydrostatic axis separates the two theories. For the special cases of plane stress and plane strain and considering c as a constant, the above coincidence has earlier been shown by Clavout and Ziegler [12].

The choice between Melan-Prager's and Ziegler's hardening rule should therefore be based on mathematical convenience, and here Ziegler's hardening rule seems to have some advantages.

SOME NON-LINEAR HARDENING FUNCTIONS

The following discussion is devoted to some of the proposals made for the hardening function c present in eq.(6). To obtain an interpretation of c, multiply eq.(5) by $d\lambda$ and use eq.(3)

$$d\varepsilon_{ij}^{\ P}\, d\varepsilon_{ij}^{\ P} = \frac{1}{c}\, d\varepsilon_{kl}^{\ P}\, d\sigma_{kl}$$

For uniaxial tensile loading using the condition of plastic incompressibility, the above equation implies

$$\frac{d\sigma_{11}}{d\varepsilon_{11}^{\ P}} = \frac{3}{2}\, c \tag{20}$$

where direction 11 corresponds to the direction of the tensile loading.

In the classical linear hardening theory of Melan [4] and Prager [2], [3], c is considered as a constant corresponding to bilinear stress-strain curves. The first proposal for non-linear hardening seems to be that made by Kadashevish and Novozhilov [14], who assumed that

$$\alpha_{ij} = g\, \varepsilon_{ij}^{\ P} \tag{21}$$

corresponding to eq.(9), but where the symbol g is used instead of c, as g is assumed to be the following function

$$g = g[(\alpha_{ij}\ \alpha_{ij})^{\frac{1}{2}}]$$

where the invariant $(\alpha_{ij}\ \alpha_{ij})^{\frac{1}{2}}$ is the distance OC in fig. 1. Now, eq.(21) and the above equation are a special case of

$$\alpha_{ij} = g(\alpha_{ij})\ \varepsilon_{ij}^{P} \tag{22}$$

and as this equation is assumed to state a unique relationship between α_{ij} and ε_{ij}^{P}, it is equivalent to

$$\alpha_{ij} = h(\varepsilon_{ij}^{P})\ \varepsilon_{ij}^{P} \tag{23}$$

Following Eisenberg and Phillips [15], we then consider the uniaxial stress-plastic strain curve of fig. 2.

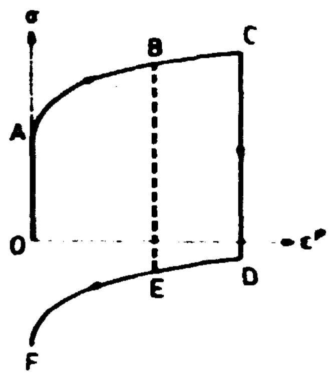

Fig. 2.

As a result of eq.(2) and (23) it appears that $|BE| = |CD| = 2|AO|$, i.e. the curve FED is congruent to ABC. It is obvious that fig. 2 is not representative of the behaviour of metals, and the proposals of eqs.(22) and (23), which are generalizations of the proposal of Kadashevish and Novozhilov [14], are therefore unsatisfactory.

In [11] Ziegler noted, without going into details, that c could be considered as a function of the distance OC in fig. 1, i.e.

$$c = c[(\alpha_{ij}\ \alpha_{ij})^{\frac{1}{2}}] \tag{24}$$

or as a function of the plastic work W_p, i.e.

$$c = c(W_p) \tag{25}$$

where the differential of the plastic work is defined as

$$dW_p = \sigma_{ij}\, d\varepsilon_{ij}^{\,p}$$

However, it is evident that eq.(24) implies the same unrealistic reversed curve for uniaxial loading as shown in fig. 2. If eq.(25) is assumed, and if it is further assumed that the origin 0 in fig. 1 is located inside any loading surface during loading, then W_p is an increasing function, and fig. 3 hence shows the implications of reversed uniaxial loading.

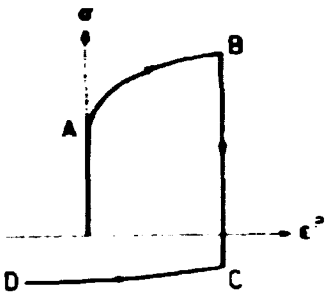

Fig. 3.

The slopes at B and C in fig. 3 are identical; and the slope steadily decreases along CD, if the slope on the unreversed curve beyond B also decreases.

However, if eq.(25) is again accepted, but it is now assumed that during hardening the origin 0 of fig. 1 is no longer located inside the loading surface, then W_p can decrease during reversing. Following Hunsaker et al. [16], the reversed uniaxial loading then takes the form of fig. 4, where the stress-strain curve, because of simplicity, is assumed to consist of straight lines.

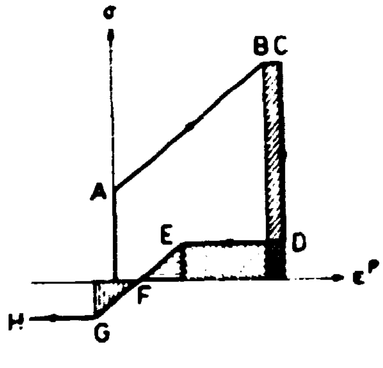

Fig. 4.

The slopes at C and D in fig. 4 are identical, and negative plastic work is performed along DE. At E this negative plastic work has cancelled the positive plastic work done from B to C, and the slope along EF then becomes identical to the slope along AB. Further negative plastic work is performed along EF, and when the positive plastic work done along FG eliminates this work, the slope along GH becomes identical to the slope along BC.

Thus, even though the behaviour shown in fig. 3 has a certain resemblance to actual metal behaviour, the behaviour indicated in fig. 4 is very unrealistic, so that the assumption of eq.(25) cannot be justified.

The proposal of Isakson et al. |17| makes use of the Ramberg-Osgood approximation |18| to the uniaxial stress-strain curve

$$\varepsilon = \frac{\sigma}{E} + \frac{3\sigma}{7E} \left|\frac{\sigma}{\sigma_{0.7}}\right|^{n-1} \tag{26}$$

where ε is the total strain, and the shape parameter n ($n>1$) is given by

$$n = 1 + \frac{\ln \frac{17}{7}}{\ln \frac{\sigma_{0.7}}{\sigma_{0.85}}}$$

E denotes Young's modulus, while $\sigma_{0.7}$ and $\sigma_{0.85}$ are the stresses at which the secant moduli are 0.7E and 0.85 E, respectively. From eq.(26) we obtain

$$\frac{d\varepsilon^p}{d\sigma} = \frac{3n}{7E} \left|\frac{\sigma}{\sigma_{0.7}}\right|^{n-1} \tag{27}$$

If eq.(27) is written for every stress and strain component, then a typical form is

$$\frac{d\varepsilon_{xy}^{\ p}}{d\sigma_{xy}} = \frac{3n_{xy}}{7E_{xy}} \left|\frac{\sigma_{xy}}{\sigma_{xy,0.7}}\right|^{n_{xy}-1} \tag{28}$$

where, for instance, the modulus E_{xy} is defined by the linear elastic relation $\sigma_{xy} = E_{xy}\, \varepsilon_{xy}$. The notation in eq. (28) is then obvious as, for instance, subscripts xy and xx correspond to shear loading and uniaxial loading, respectively. Hence, six parameters correspond to all possible forms of eq.(28). These parameters are derivable from the experimental uniaxial and shear loading, and Isakson et al. [17], treating only plane stress, proposed the following hardening function

$$\frac{1}{c} = \frac{d\sigma_{xx}}{d\varepsilon_{xx}^{\ p}}\left(\frac{\sigma_{xx}}{\bar{\sigma}}\right)^2 + \frac{d\sigma_{yy}}{d\varepsilon_{yy}^{\ p}}\left(\frac{\sigma_{yy}}{\bar{\sigma}}\right)^2 + \frac{d\sigma_{xy}}{d\varepsilon_{xy}^{\ p}}\left(\frac{\sigma_{xy}}{\bar{\sigma}}\right)^2 \tag{29}$$

where $\bar{\sigma}$ is defined by

$$\bar{\sigma} = (\sigma_{xx}^{\ 2} + \sigma_{yy}^{\ 2} + \sigma_{xy}^{\ 2})^{\frac{1}{2}}$$

and eq.(28) is applied. Noting that Isakson et al. treated c as identical to the slope of the stress-plastic strain curve without involving the factor 3/2 present in eq.(20), it appears that eq.(29) gives the correct c-values for non-reversed uniaxial and shear loading, but as noted by Armen et al. [19], eq.(29) is not invariant with respect to a rotation of the coordinate axes, and if eq.(29) is generalized to multiaxial stress states, c becomes dependent on hydrostatic stress.

Regarding reversed loading, Isakson et al. [17] made the important assumption that the inelastic parts of the curves after reversed loading are identical to those that follow during the initial yielding. This assumption is illustrated in fig. 5 for uniaxial loading.

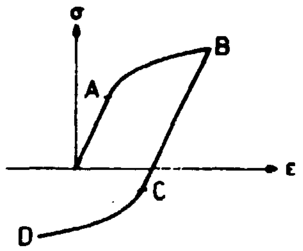

Fig. 5.

Here curve CD is congruent to curve AB, and consequently the hysteresis curve for zero mean stress or strain is point symmetric around the origin in fig. 5; this behaviour is in fact the actual experimental behaviour in the steady-state stage, Krempl [20]. The particular case under consideration with completely reversed multiaxial loading is indicated in fig. 6, where 0 is the origin of the stress space, and loading 0A is followed by loading AB.

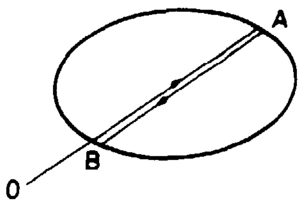

Fig. 6.

Isakson et al. [17] noted that the above assumption can be represented by replacing σ_{xx} by $\sigma_{xx}-\alpha_{xx,A}$ in eq.(29) and analogously for the other stress components where $\alpha_{xx,A}$ denotes the value of α_{xx} at point A.

Also Eisenberg and Phillips [15] made use of the assumption of eq.(21), but they assumed that

$$g = g(\kappa_1)$$

where the differential of κ_1 is defined by

$$d\kappa_1 = (d\varepsilon_{ij}^{P}\, d\varepsilon_{ij}^{P})^{\frac{1}{2}}$$

i.e. κ_1 is a steadily increasing function, and the case of reversed uniaxial loading is therefore shown in fig. 3 - cf. the discussion of eq.(25).

Armen et al. [19], who just as Isakson et al. [17] treated c as identical to the slope of the uniaxial stress - plastic strain curve, assumed by analogy to eq.(27) that

$$\frac{1}{c} = \frac{3n}{7E} \left|\frac{\sigma_e}{\sigma_{0.7}}\right|^{n-1} \qquad (31)$$

where the effective stress σ_e for plane stress was defined by

$$\sigma_e = (\sigma_{xx}^{\;2} - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^{\;2} + 3\,\sigma_{xy}^{\;2})^{\frac{1}{2}}$$

in accordance with the von Mises criterion. The assumption of eq.(31) has the consequences of giving the correct c-value for uniaxial loading, and it implies that c depends on the location of the stress point on the loading surface. However, use of eq.(31) is restricted to a localized region on the loading surface, as no general criterion for the occurrence of reversed loading was stated, and only the completely reversed loading shown in fig. 6 was treated in a way similar to that of Isakson et al. [17]. Levine and Svalbonas [21] showed a generalization of eq.(31) to account for initially orthotropic material behaviour.

To approximate the cyclic hardening behaviour of metals, Pugh et al. [22], using the classical linear representation, suggested a general criterion stating that reversed loading is initiated if

$$\alpha_{ij}\;\; d\varepsilon_{ij}^{P} \leq 0 \qquad (32)$$

I.e., reversed loading occurs when the stress point in fig. 7 falls outside

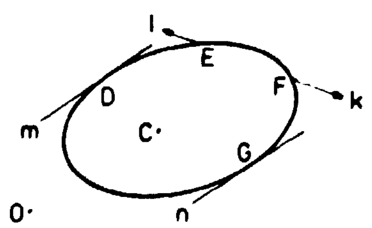

Fig. 7.

the part DEFG of the loading surface, whose centre is C. O is the origin of the stress space, and lines m and n are parallel to OC. However, eq.(32) is only applicable tc determine the initiation of one reversed loading, for instance that of point B in fig. 6, which followed loading at point A, but when loading is again reversed from B to A, this is not traced by eq.(32).

To consider this effect, eq.(32) can be replaced by

$$\alpha^{*}_{ij}\, d\varepsilon_{ij}{}^{p} \leq 0 \tag{33}$$

where

$$\alpha^{*}_{ij} = \int_{X_1}^{X} d\alpha_{ij} \tag{34}$$

X corresponds to the actual stress point, while X_1 corresponds originally to the initiation of plastic behaviour and later to the situation where the last reversed loading is initiated. Figure 8 illustrates the integration path for two cases of uniaxial loading.

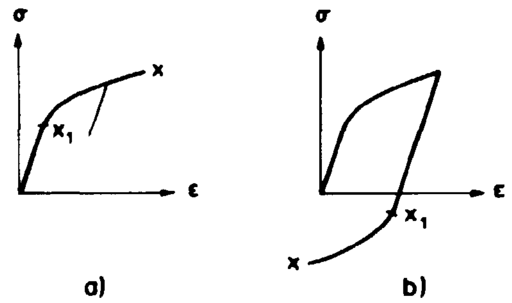

Fig. 8.

Fig. 8a) shows that elastic unloading and reloading do not influence the integration path, while 8b) shows the case of reversed loading.

Eq.(34) was in fact proposed by Rashid [23], who instead of eq.(33) used the following definition for the initiation of reversed loading

$$\alpha^{*}_{ij}\, d\sigma_{ij} < 0 \tag{35}$$

together with the condition that plastic deformation occurs, i.e. $d\sigma_{ij}$ $\partial f/\partial\sigma_{ij} > 0$. The superiority of eq.(35) compared to eq.(33) is illustrated in fig. 7, where loading at point F in direction k followed by loading at point E in direction 1 is not considered as reversed loading when eq.(33) is used, while eq.(35) predicts reversed loading. Combined with eqs.(34) and (35) Rashid [23] made the following assumption for Melan-Prager's hardening rule - compare also eq.(24)

$$c = c[(\alpha^*_{ij}\ \alpha^*_{ij})^{\frac{1}{2}}]$$

Reversed uniaxial loading as predicted by this equation is shown in fig. 5, and even though Rashid's theory has considerable advantages, it has some disadvantages too, in particular a discontinuous behaviour, which seems unrealistic, follows in some loading cases as shown in fig. 9. Here loading at point A is followed by loading at point B, where line l is perpendicular to OC.

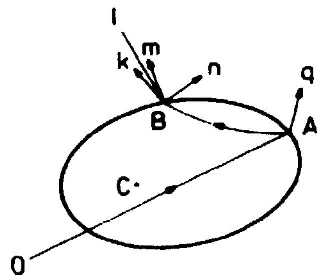

Fig. 9.

Loading at B in directions m and n gives the same c-value as loading at A in direction q, while loading at B in direction k corresponds to reversed loading and thus a completely different c-value.

FORMULATION OF A NEW NON-LINEAR HARDENING FUNCTION

In general, the hardening function c depends on the loading history and the present loading. These requirements together with a smooth change in the behaviour from one plastic state to another nearlying plastic state and consideration of the problem of general reversed loading can be fulfilled by adopting eq.(34) together with the criterion for initiation of reversed loading, eq.(35), and defining the tensor $\bar{\alpha}_{ij}$ by

$$\bar{\alpha}_{ij} = \alpha^*_{ij} \frac{\alpha^*_{kl}}{(\alpha^*_{st}\ \alpha^*_{st})^{\frac{1}{2}}} \frac{d\sigma_{kl}}{(d\sigma_{mn}\ d\sigma_{mn})^{\frac{1}{2}}} \tag{36}$$

The hardening function c, applicable both for Melan-Prager's and for Ziegler's hardening rule, is then determined by the following assumption

$$c = c[(\bar{\alpha}_{ij}{}^D\ \bar{\alpha}_{ij}{}^D)^{\frac{1}{2}}] \tag{37}$$

where the deviatoric part $\bar{\alpha}_{ij}{}^D$ of the tensor $\bar{\alpha}_{ij}$ is defined by

$$\bar{\alpha}_{ij}{}^D = \bar{\alpha}_{ij} - \frac{1}{3}\delta_{ij}\bar{\alpha}_{kk}$$

For proportional loading and for the completely reversed loading shown in principle in fig. 6, eq.(36) implies that $\bar{\alpha}_{ij} = \overset{*}{\alpha}_{ij}$. For non-proportional loading, the factor $\overset{*}{\alpha}_{kl}\, d\sigma_{kl}/[(\overset{*}{\alpha}_{st}\,\overset{*}{\alpha}_{st})^{\frac{1}{2}}\,(d\sigma_{mn}\, d\sigma_{mn})^{\frac{1}{2}}]$ present in eq.(36) and situated between zero and unity, lowers $\bar{\alpha}_{ij}$ compared to $\overset{*}{\alpha}_{ij}$, whereby a completely smooth behaviour is assured, when going from proportional loading to loading directions more and more distant from it. It appears that eq.(37) implies a reversed uniaxial stress-strain curve as shown in fig. 5.

To determine the hardening function c defined by eq.(37) for a given material, some calibration test is required, the most convenient one being of course the uniaxial tensile test. Then, considering eq.(12), the use of Melan-Prager's hardening rule, where $\alpha_{ij} = \alpha_{ij}{}^D$ - cf. eq.(19), and also the use of Ziegler's hardening rule, where the only non-zero component of the tensor α_{ij} is α_{11}, corresponding to the direction of the tensile loading, implies that

$$\sigma_{11} = \sigma_0 + (\frac{3}{2}\bar{\alpha}_{ij}{}^D\,\bar{\alpha}_{ij}{}^D)^{\frac{1}{2}} \tag{38}$$

Use of this equation together with eqs.(20) and (37) makes the determination of the hardening function c possible, once the uniaxial stress-strain curve is known.

If, for instance, the Ramberg-Osgood approximation [18] to the uniaxial stress-strain curve, eq.(26), is adopted, then we obtain

$$c = \frac{14E}{9n}\left|\frac{\sigma_{0.7}}{\sigma_0 + (\frac{3}{2}\bar{\alpha}_{ij}{}^D\,\bar{\alpha}_{ij}{}^D)^{\frac{1}{2}}}\right|^{n-1} \tag{39}$$

Another useful and completely smooth approximation to the uniaxial stress-strain curve given by Barnard and Sharman [24] is

$$\sigma \leq \sigma_0 \qquad \varepsilon^P = 0$$

$$\sigma \geq \sigma_0 \qquad \varepsilon^P = A\,(\sigma - \sigma_0)^B$$

where A and B are parameters. The hardening function becomes

$$c = \frac{2}{3AB}(\frac{3}{2}\bar{\alpha}_{ij}{}^D\,\bar{\alpha}_{ij}{}^D)^{-(B-1)/2} \tag{40}$$

Through the above determination of the hardening function c together with the equations (12), (15), (17) and (18), all information is available for an elastic-plastic analysis. However, to present the results in a more advantageous form, notably for use in finite element analysis, the procedure introduced by Yamada et al. [25] is followed below in the form given by Zienkiewicz et al. [26]. If we use eq. (5) and Hooke's law

$$d\sigma_{ij} = D_{ijkl}\,(d\varepsilon_{kl} - d\varepsilon_{kl}{}^{p})$$

where ε_{ij} is the total strain tensor, and the elasticity tensor is denoted by D_{ijkl}, then we obtain

$$d\sigma_{ij} = D_{ijkl}{}^{p}\, d\varepsilon_{kl} \tag{41}$$

where the plasticity tensor $D_{ijkl}{}^{p}$, possessing the same properties of symmetry as D_{ijkl}, is

$$D_{ijkl}{}^{p} = D_{ijkl} - D_{ijmn}\, A_{mnkl}$$

where

$$A_{mnkl} = \frac{D_{stkl}\,\dfrac{\partial f}{\partial \sigma_{st}}\,\dfrac{\partial f}{\partial \sigma_{mn}}}{\dfrac{3}{2}\,c + D_{stmn}\,\dfrac{\partial f}{\partial \sigma_{st}}\,\dfrac{\partial f}{\partial \sigma_{mn}}}$$

Besides, we find

$$d\varepsilon_{ij}{}^{p} = A_{ijkl}\, d\varepsilon_{kl} \tag{42}$$

As we restrict ourselves to metals possessing an initial isotropy, then

$$D_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + G\,(\delta_{ik}\, \delta_{jl} + \delta_{jk}\, \delta_{il}) \tag{43}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)} \qquad \text{and} \qquad G = \frac{E}{2(1+\nu)}$$

are Lame's constants. E and ν are Young's modulus and Poisson's ratio, respectively. By means of eqs.(43) and (13), we obtain the following form of eq.(41)

$$d\sigma_{ij} = \left(D_{ijkl} - \frac{9G^2}{\sigma_0{}^2\,(\frac{3}{2}\,c + 3\,G)}\, s_{ij}'\, s_{kl}'\right) d\varepsilon_{kl} \tag{44}$$

and eq.(42) becomes

$$d\varepsilon_{ij}{}^{p} = \frac{9G}{2\sigma_0{}^2\,(\frac{3}{2}\,c + 3\,G)}\, s_{ij}'\, s_{kl}'\, d\varepsilon_{kl} \tag{45}$$

Combination of eqs.(3) and (45) implies that

$$d\lambda = \frac{3G}{\sigma_0^{\,2}\,(\frac{3}{2}c + 3\,G)}\; s_{kl}'\, d\varepsilon_{kl} \tag{46}$$

It should be emphasized that eqs.(44), (45), and (46) still apply to ideal plasticity where $c = 0$. For Melan-Prager's hardening rule, the equations necessary for an elastic-plastic analysis are given by eqs.(44), (45), and (8), while the corresponding equations using Ziegler's hardening rule are given by eqs.(44), (11), (14), and (46). In both cases these equations should be coupled with eqs.(12), (34), (35), (36), and, for instance, eqs.(39) or (40). It should be noted that due to the anisotropy induced by α_{ij}, it is, contrary to isotropic hardening, possible to work only with principal stresses and strains provided that the principal axes of the stress tensor do not rotate, as only in this case do the principal axes of the stress tensor and the strain tensor coincide. This is easily shown by means of eqs.(44) and (45), and was originally shown by Shield and Ziegler [10] for Melan-Prager's hardening rule and by Ziegler [11] for his own hardening rule.

It will now be shown that kinematic hardening and isotropic hardening give identical results for proportional loading, provided that the same uniaxial stress-strain curve is used for the calibration of the hardening function.

It is easily shown, e.g. Donea et al. [27], that for isotropic hardening eqs.(44), (45), and (46) still apply if s_{ij}' and σ_0 are replaced by s_{ij} and σ_e, where σ_e is the effective stress defined by - cf. eq.(12)

$$\sigma_e = (\tfrac{3}{2}\, s_{ij}\, s_{ij})^{\frac{1}{2}}$$

Besides, the term $3c/2$ should be replaced by $d\sigma_e/d\varepsilon_e^{\,p}$, where the differential of the equivalent plastic strain is defined by

$$d\varepsilon_e^{\,p} = (\tfrac{2}{3}\, d\varepsilon_{ij}^{\,p}\, d\varepsilon_{ij}^{\,p})^{\frac{1}{2}}$$

As we only consider increasing proportional loading, we have

$$\left(\frac{s_{ij}}{\sigma_e}\right)_{\text{isotropic}} = \left(\frac{s_{ij}'}{\sigma_0}\right)_{\text{kinematic}}$$

i.e. kinematic hardening and isotropic hardening give identical results, if the terms $3c/2$ and $d\sigma_e/d\varepsilon_e^{\,p}$ are identical for a given stress state. But this is certainly true, as both terms correspond to the slope at the same point on the uniaxial stress-plastic strain curve, which was adopted for calibration, because the following unique relation exists for proportional loading, namely

$$\sigma_e = \sigma_0 + (\frac{3}{2}\bar{\alpha}_{ij}{}^D \bar{\alpha}_{ij}{}^D)^{\frac{1}{2}}$$

Now, eqs.(44), (45), and (46) apply in the general stress state, and it is advantageous to consider the form of these equations for the particular cases of axial symmetry, generalized plane strain and plane stress.

Axial symmetry. By definition we have, for instance,

$$\varepsilon_{13} = \varepsilon_{23} = \sigma_{13} = \sigma_{23} = \alpha_{13} = \alpha_{23} = 0 \tag{47}$$

which implies

$$s_{13}' = s_{23}' = \varepsilon_{13}{}^p = \varepsilon_{23}{}^p = 0 \tag{48}$$

Hence, eqs.(44), (45), and (46) can be used directly simply noting eqs.(47) and (48), and the only directions of interest are 11, 22, 33, 12.

Generalized plane strain. By definition we have, for instance,

$$\varepsilon_{33} = \text{constant};\ \varepsilon_{13} = \varepsilon_{23} = 0 \tag{49}$$

which implies

$$\sigma_{13} = \sigma_{23} = \alpha_{13} = \alpha_{23} = \varepsilon_{13}{}^p = \varepsilon_{23}{}^p = s_{13}' = s_{23}' = 0 \tag{50}$$

To prove, for instance, $\sigma_{13} = 0$ given by eq.(50), consider eqs.(44) and (45) which imply that $d\sigma_{13}$ and $d\varepsilon_{13}{}^p$ are proportional to $s_{13}' = \sigma_{13} - \alpha_{13}$. When only elastic deformations occur, then $\sigma_{13} = \alpha_{13} = 0$, and at the moment when plastic deformations are initiated it follows that $d\sigma_{13} = d\varepsilon_{13}{}^p = s_{13}' = 0$, i.e. $\sigma_{13} = \varepsilon_{13}{}^p = s_{13}' = 0$ applies in general. Therefore, both for Melan-Prager's and for Ziegler's hardening rule, we have $\alpha_{13} = 0$, which completes the existence of eq.(50).

Now, calculation of, for instance $d\sigma_{11}$ requires knowledge of σ_{33} and α_{33} through the existense of s_{11}', and therefore it is necessary to keep a record of $d\sigma_{33}$ and for Melan-Prager's hardening rule also of $d\varepsilon_{33}{}^p$. Hence, eqs.(44), (45), and (46) can be used directly simply noting eqs.(49) and (50), and the directions of interest are 11, 22, 33 and 12. This is contrary to elastic plane strain, where only directions 11, 22 and 12 need to be considered.

Plane stress. By definition we have, for instance,

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \tag{51}$$

which implies

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{13}^{p} = \varepsilon_{23}^{p} = \alpha_{13} = \alpha_{23} = s_{13}' = s_{23}' = 0 \tag{52}$$

To prove, for instance, $\varepsilon_{13} = 0$ given by eq.(52), consider eq.(45), which shows that $d\varepsilon_{13}^{p}$ is proportional to $s_{13}' = \alpha_{13}$. Thus, at the moment when plastic deformations are initiated, we have $d\varepsilon_{13}^{p} = 0$ and therefore $d\alpha_{13} = 0$ both for Melan-Prager's and for Ziegler's hardening rule, i.e. $s_{13}' = \alpha_{13} = \varepsilon_{13}^{p} = 0$ applies in general. Equation (44) then shows that also $\varepsilon_{13} = 0$ applies in general, which completes the existence of eq.(52).

Now, for plane stress, it is advantageous to use matrix notation, and eq.(44) in combination with eqs.(51) and (52) implies

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{12} \\ 0 \end{Bmatrix} = [P] \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \\ d\varepsilon_{33} \end{Bmatrix} \tag{53}$$

where $[P]$ is a symmetric 4 x 4 matrix. Elimination of $d\varepsilon_{33}$ from eq.(53) is possible by static condensation, whereby we obtain

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{Bmatrix}$$

where $[Q]$ is a symmetric 3 x 3 matrix given by

$$Q_{ij} = P_{ij} - \frac{P_{i4}\,P_{j4}}{P_{44}}$$

with obvious notation. Besides we obtain

$$d\varepsilon_{33} = -\frac{1}{P_{44}}\{P_{41}, P_{42}, P_{43}\} \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{Bmatrix}$$

which shows that also all the non-zero components of $d\varepsilon_{ij}^{p}$ given by eq.(45) and $d\lambda$ given by eq.(46) can be obtained using only $d\varepsilon_{11}$, $d\varepsilon_{11}$ and $d\varepsilon_{12}$. Finally, for Ziegler's hardening rule, we have $\alpha_{33} = 0$.

CONCLUSION AND DISCUSSION

Based on the classical theory of plasticity and accepting the von Mises criterion as the initial yield criterion, a non-linear kinematic hardening function applicable both to Melan-Prager's, eq.(8), and to Ziegler's hardening rule, eq.(11), is proposed. This non-linear hardening function defined by eq.(37) can be used in connection with any uniaxial stress-strain curve and takes into account the problem of general reversed loading by means of eqs. (36), (35), and (34). One consequence of these assumptions is that for the reversed uniaxial loading of fig. 5, curve CD is congruent to curve AB, implying a hysteresis curve for zero mean stress or strain that is point symmetric around the origin in fig. 5 in accordance with experimental facts for metals in the cyclic steady-state stage, Krempl [20]. Another consequence is the fulfilment of a completely smooth behaviour, when going from proportional loading to loading directions more and more distant from it. It is also shown that material behaviour is identical whether Melan-Prager's or Ziegler's hardening rule is applied, provided that the von Mises yield criterion is adopted. Besides, for increasing proportional loading, application of isotropic or kinematic hardening implies the same results provided that the same uniaxial stress-strain curve is used for the calibration of the hardening function, and that the von Mises criterion is applied.

To present the results in a form convenient in particular to finite element analysis, the explicit relations between $d\sigma_{ij}$ and $d\varepsilon_{ij}$ and between $d\varepsilon_{ij}^{p}$ and $d\varepsilon_{ij}$ are given in eqs.(44) and (45), respectively, together with the explicit expression, eq.(46), for $d\lambda$, all valid for ideal plasticity too. Similar relations are also given for the cases of axial symmetry, generalized plane strain and plane stress. In addition, explicit expressions, eqs. (39) and (40), are given for the hardening function when the uniaxial stress-strain curve is approximated by the Ramberg-Osgood formula [18] and the formula given by Barnard and Sharman [24], respectively.

Even though the above proposals represent a considerable extension of the classical linear hardening theory, they do not reproduce the special aspects of metal behaviour connected with cyclic hardening and softening, but in principle such phenomena could be described by a theory combining kinematic and isotropic hardening. It should also be noted that the above proposal for a non-linear kinematic hardening function could be generalized in a straightforward manner to include non-isothermal behaviour, where the initial uniaxial yield stress σ_0 is temperature dependent, following the lines of Chang [28], who extended the work of Prager [29] for rigid-plastic materials to elastic-plastic materials.

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