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Holographic reconstruction of sound fields based on the acousto-optic effect

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ABSTRACT

Recent studies have shown that it is possible to measure a sound field using acousto-optic tomography. The acousto-optic effect, i.e., the interaction between sound and light, can be used to measure an arbitrary sound field by scanning it with a laser Doppler vibrometer (LDV) over an aperture; This can be described mathematically by means of the Radon transform of the acoustic field. An interesting feature of this measurement technique is that the spatial characteristics of the sound field are captured in the measurement. Therefore, the technique has an inherent holographic potential, implicitly yielding a full characterization of the sound field. In this study, a direct projection of the Radon transform from one plane to another and into the space domain, based on an elementary wave expansion is proposed. The relationship between the Radon and the wavenumber domains is examined, and the reconstruction potential of the method analyzed. The study includes both numerical and experimental results.

Keywords: Near-field acoustic holography (NAH); Sound visualization; Acousto-optic effect

1. INTRODUCTION

Sound source identification and sound visualization methods are a very useful tool for studying the sound radiated by acoustic sources. They make it possible to localize and quantify the sound radiated by a source, or parts of a source, and study the mechanisms that produce the acoustic radiation based on measurements in-situ. Consequently, they are a useful tool in the design of acoustic sources, or for reducing their noise output. Some of the most popular methods are based on measurement with arrays of transducers, such as beamforming\textsuperscript{1} and near-field acoustic holography (NAH).\textsuperscript{2–4}

Recent studies\textsuperscript{5–7} have shown that it is possible to measure the sound pressure over an aperture via tomographic measurements based on the acousto-optic effect, i.e., the interaction between sound and light.
Using a laser Doppler vibrometer (LDV), it is possible to scan the sound field over an area and obtain the Radon transform of the sound pressure. From these data, the actual pressure in space domain can be retrieved via the corresponding inverse transform.

This measurement principle has an inherent holographic potential, which is examined in this paper. The potential stems from he fact that the amplitude and phase of the sound waves is measured over an area, thus it is possible to predict the sound field in a different plane via a holographic reconstruction. This is closely related to acoustic holography and NAH, except that the starting point is not the sound pressure sampled with an array of microphones, but the Radon transform obtained from the measurement of the sound pressure via the acousto-optic effect with a laser Doppler vibrometer. The holographic reconstruction from the Radon transform can be based on an explicit spatial Fourier transformation of the measured field (with FFT), as in the original NAH processing. This possibility was recently examined, and is in a way analogous to the filtered back-projection method, except that the wavenumber spectrum is propagated from one plane to another prior to the inverse transformation yielding the pressure in a different plane.

In this paper a reconstruction technique is proposed which directly projects the Radon transform from one plane to another and into space domain by means of an elementary plane wave expansion. This is in a way similar to the Fourier based reconstruction, only that in this case the solution of the system does not rely on explicit Fourier transformations, and thus some of the limitations related to the DFT are avoided or minimized, most notably wraparound error and truncation errors related to the finite aperture.

The present study examines the relationship between the Radon domain and the wavenumber domain, the reconstruction based on the plane wave expansion, and the potential of the technique. Numerical results are shown, and a simple experimental measurement is included.

2. THEORY

A. The acousto-optic effect and tomography

The acousto-optic effect describes the interaction between sound and light, when the refractive index of the medium is changed by the presence of sound waves, which inherently involve pressure fluctuations, and consequently density changes. In the audible frequency range, this phenomenon can simply be understood as a phase modulation of the light that propagates through the acoustic field, or in other words, the light travels slightly faster/slower when the sound pressure decreases/increases. This small effect can be measured with a laser Doppler vibrometer (LDV) by letting the beam of light travel through the medium where acoustic waves are present, and focusing the beam on a steady surface. An important requisite for measuring the acousto-optic effect is that the reflecting surface remains still, otherwise the LDV will also sense mechanical vibrations that will compromise the measurement. It can be shown that the apparent velocity measured with the LDV due to the acousto-optic effect is

\[ v_{\text{LDV}}(t) = \frac{n_0 - 1}{\gamma p_0 n_0} \frac{d}{dt} \left( \int_L p(x, y, z, t) \, dl \right), \]

where \( n_0 \) and \( p_0 \) are the static refractive index and pressure of the medium under static conditions, \( \gamma \) the ratio of specific heats, and the integral of \( p(x, y, z, t) \) corresponds to the projection of the sound pressure along the path \( L \) followed by the laser beam. This is further explained in the next subsection.

It is important to note that the line integral in Eq. (1) can be used to estimate the Radon transform of the sound pressure. If the sound field is scanned as shown in Fig. 1, the Radon transform can be retrieved from the apparent velocity of the LDV as

\[ R(x', \phi, z_h, \omega) = \frac{\gamma p_0 n_0}{n_0 - 1} \frac{V_{\text{LDV}}(\omega)}{j \omega}, \]

where \( R \) is the Radon transform of the sound pressure in the frequency domain, \( V_{\text{LDV}} \) is the apparent velocity of the LDV also expressed in the frequency domain, \( \phi \) is the rotation angle of each scan, which is taken for a set of parallel lines along the \( x' \) axis, and \( z_h \) is the measurement plane. Note that we consider now the stationary harmonic case, with the convention \( e^{j \omega t} \).

B. Relation between the Radon and the wavenumber domain

In acoustics, it is common to use the wavenumber transform to analyze the spatial characteristics of a sound field, which has proven to be a powerful experimental tool used in combination with array measurements. The Radon transform is another integral transform, commonly used in tomography, that results from
integrating a function over straight lines. Nonetheless, the Radon transform also yields a spatial characterization of the sound field, in a way similar to the wavenumber domain.

Given a certain sound field in a plane \( p(x, y) \), the tomographic measurement entails scanning the field in a series of projections. We consider here the scanning based on parallel projections over a series of angles \( \phi \) from 0 to \( \pi \), as shown in Fig. 1. This operation is expressed mathematically as the Radon transform of the sound field (disregarding the time dependency),

\[
R(x', \phi) = \int_{-\infty}^{\infty} p(x, y) dy',
\]

which expresses the fact that the sound field \( p(x, y) \) is integrated along a line \( y' \) (see Fig. 1). This \((x', y')\) coordinate system used for the Radon transform is simply a rotated version of the original \((x, y)\), as expressed by the rotation matrix

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos(\phi) & \sin(\phi) \\
-\sin(\phi) & \cos(\phi)
\end{bmatrix} \cdot \begin{bmatrix}
x \\
y
\end{bmatrix},
\]

so that \( R(x', \phi) \) corresponds to the values of the projections of the sound field along \( y' \) for each projection angle \( \phi \) across the \( x' \) coordinate.

The well-known ‘Fourier slice theorem’ or ‘projection-slice theorem’ reveals the relation between the Radon and the wavenumber spectrum \( P(k_x, k_y) \), where the Fourier transform of each of the projections corresponds to a diagonal in the wavenumber spectrum, as illustrated in Fig. 2. This important relationship is well documented in the literature.\(^9\)

In the present study we approach this relation from a somewhat different perspective. The spatial domain can be expressed in polar coordinates, \( p(r, \theta) \equiv p(x, y) \), where \( x = r \cos(\theta) \), \( y = r \sin(\theta) \), so that the rotation coordinates become \( x' = r \cos(\theta - \phi) \), \( y' = r \sin(\theta - \phi) \). The Jacobian of the transformation yields \( dx dy = dx' dy' = r dr d\theta \). Equation (5) can thus be expressed as

\[
P(k_{x'}, \phi) = \int_{0}^{2\pi} \int_{0}^{\infty} p(r, \theta) e^{ik_{x'}x'} r dr d\theta.
\]

Additionally, if the wavenumber domain is expressed in polar coordinates \( (P(k_r, \phi_k) \equiv P(k_x, k_y)) \), with
Fig. 2 – Sketch of the Fourier or projection-slice theorem

\[ k_x = k_r \cos(\phi_k) \quad \text{and} \quad k_y = k_r \sin(\phi_k), \]

the expression for the wavenumber transform becomes naturally

\[
P(k_r, \phi_k) = \int_{-\infty}^{\infty} p(x, y) e^{j(k_x x + k_y y)} dx dy = \int_{0}^{2\pi} \int_{0}^{\infty} p(r, \theta) e^{j k_r r \cos(\theta - \phi_k)} r dr d\theta.
\] (7)

This shows the relationship between the Radon and wavenumber domains. It is obvious from these last two equations (6-7) that the rotation angle in the Radon transform \( \phi \) corresponds to the polar angle of the wavenumber domain \( \phi_k \). However, the projection angle \( \phi \) will span from 0 to \( \pi \), but when expressing it as \( \phi_k \) (the Radon transform in polar coordinates) it spans from 0 to \( 2\pi \) \( (\phi_k = \phi + (\pi/2)(\text{sgn}(k_{x'})) - 1) \). Similarly, \( k_{x'} \) and \( k_r \) are related by \( k_r = |k_{x'}| \), with \( -\infty < k_{x'} < \infty \) and \( 0 \leq k_r < \infty \).

**C. Holography from the Radon transform**

The fundamental principle behind holographic methods is that the entire sound field can be reconstructed in three dimensions by accounting for the propagation of the sound waves from the measurement position to another. This can be done by operating in the wavenumber domain, which can be obtained by Fourier transforming the field explicitly via FFT’s. An alternative way of operating, is using an elementary wave expansion, which avoids explicit Fourier transforms, thus several of the shortcomings and errors related to transforming the field explicitly via FFT’s.

As anticipated in the previous section, the wavenumber spectrum obtained from the Radon transform via the projection slice theorem is equivalent to the wavenumber spectrum in polar coordinates \( P(k_{x'}, \phi, z_h) \equiv P(k_r, \phi_k, z_h) \), with the difference that \( P(k_{x'}, \phi, z_h) \) is not strictly in polar coordinates since \( -\infty < k_{x'} < \infty \) and \( 0 \leq \phi \leq \pi \), whereas for \( P(k_r, \phi_k, z_h) \) the coordinates satisfy \( 0 < k_r < \infty \) and \( 0 \leq \phi_k \leq 2\pi \). In the following we will express the wavenumber spectrum in these pseudo-polar coordinates as \( P(k_{x'}, \phi, z_h) \).

In acoustic holography it is possible to extrapolate the acoustic field from one plane to another by propagating the waves present in the hologram plane \( z_h \) to the reconstruction plane \( z_s \),

\[
P(k_{x'}, \phi_k, z_h) = P(k_{x'}, \phi_k, z_h) \cdot e^{-jk_s(z_s - z_h)},
\] (10)
where

\[
k_z = \begin{cases} \sqrt{k^2 - k_{x'}^2} & \text{if } k \geq |k_{x'}| \\ \sqrt{k^2 - k_{z'}^2} & \text{if } k < |k_{x'}|, \end{cases}
\] (11)

indicates the propagating or evanescent waves in the \( z \) direction. It follows that after calculating the wavenumber spectrum from the Radon transform, it is possible to reconstruct or predict the sound field in a different plane than measured, \( z = z_s \) as

\[
p(r, \theta, z_s) = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^\infty P(k_{x'}, \phi, z_h)e^{-jk_{x'}xz'cos(\theta-\phi)+jk_z(z_h-z_s)}k_{x'}dk_{x'}d\phi.
\] (12)

Making use of Euler’s equation of motion (conservation of momentum), the particle velocity vector can be calculated, e.g. \( z \)-component, as

\[
u_z(r, \theta, z_s) = -\frac{-1}{j\omega \rho \partial \phi} = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^\infty \frac{k_z}{\rho \omega k} P(k_{x'}, \phi, z_h)e^{-jk_{x'}xz'cos(\theta-\phi)+jk_z(z_h-z_s)}k_{x'}dk_{x'}d\phi,
\] (13)

where \( \rho \) is the density of the medium and \( k \) is the wavenumber in air. It follows that it is possible to calculate the sound intensity vector too from Eqs. (12) and (13).

Finally, making use of Eqs. (8) and (12), the reconstructed sound pressure can be expressed by

\[
p(r, \theta, z_s) = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^\infty (\int_{-\infty}^{\infty} R(x', \phi, z_h)e^{-jk_{x'}xz'}dx') e^{-jk_{x'}xz'cos(\theta-\phi)+jk_z(z_h-z_s)}k_{x'}dk_{x'}d\phi.
\] (14)

### D. Discrete implementation

So far, we have considered an ideal continuous case. In practice, we implement Eq. (9) based on a discrete number of \( N \) plane waves,

\[
R(r, \phi_m) = \sum_{n=1}^{N} B_{nm} \varphi_n = \sum_{n=1}^{N} B_{nm} e^{-j(k_{x'}x'+k_z z_h)},
\] (15)

which shows the useful property that the transformation of each projection angle \( \phi_m \) depends on the \( n \) waves traveling in that direction only, as follows from the projection-slice theorem. This equation can be expressed in matrix form as

\[
r = Ab,
\] (16)

where \( r \) is a vector containing the measured Radon transform, \( b \) a vector with the corresponding coefficients (i.e., the coefficients of the wavenumber spectrum), and \( A \) is a matrix containing the elementary wave functions \( \varphi \) in the measurement plane \( z_h \), which are \( \varphi = e^{-j(k_{x'}x'+k_z z_h)} \). The pressure in the reconstruction plane can be expressed as

\[
p_s = A_s b,
\] (17)

where the matrix \( A_s \) contains the elementary waves in the reconstruction plane \( z_s \). In this case, the reconstruction matrix projects the field from the wavenumber domain into space domain, so the elementary functions depend both on \( r \) and \( \theta \),

\[
\varphi_s = e^{-j(k_{x'}x'\cos(\theta-\phi)+k_zz_s)},
\] (18)

Equation (16) can be inverted to solve for the coefficients of the expansion in a least squares sense. It should be noted that the pseudo-inversion must be regularized,\(^\text{12}\) since the back-propagation towards a sound source is an inverse ill-posed problem (due to the presence of decaying evanescent waves),

\[
b = (A^H A + \lambda I)^{-1} A^H r,
\] (19)

where \( \lambda \) is the regularization parameter (in this expression using Tikhonov regularization) and the superscript \( H \) denotes the Hermitian transpose. Making use of this equation and Eq. (17), the reconstructed pressure in the prediction plane can finally be expressed as

\[
p_s = A_s (A^H A + \lambda I)^{-1} A^H r,
\] (20)
where it becomes apparent that the measured Radon transform can be used to project directly the sound pressure into a different plane than measured. Equation (20) can be understood as a transfer function between the measured Radon transform and the predicted reconstructed pressure, where the transfer matrix \( H = A_s(A^H A + \lambda I)^{-1} A^H \) accounts for the propagation of the sound waves from the measured to the reconstruction plane and from Radon to space domain.

Similarly, as in Eq. (13), the particle velocity can be calculated by using a reconstruction matrix \( D_z \) with the derivative of the elementary wave functions with respect to the corresponding component,

\[
    u_{zs} = D_{zs}(A^H A + \lambda I)^{-1} A^H r.
\]

The elementary wave functions in the reconstruction matrix \( D_{zs} \) are

\[
    -1 j \omega \rho \frac{\partial \varphi_s}{\partial z} = \frac{k_z}{\rho c k} e^{-j(k_x x' \cos(\theta - \phi) + k_z z_h)}.
\]

The matrices in Eqs. (20) and (21) can be based on a continuous wavenumber spectrum, by using infinitely many waves expressed as an integral in the wavenumber domain. In this way the wraparound error is solved completely, since there are no replicated apertures in space domain. This is analogous to the approach suggested by Hald for statistically optimized near-field acoustic holography (SONAH).\(^{10}\)

Additionally, it is possible to pose the problem in a least norm sense, which is computationally more efficient since the characteristic dimensions of the matrices in the inversion will be smaller (there will typically be more elementary waves in the expansion than measurement positions).\(^{10}\) However, due to the projection-slice theorem, it is worth noting that the inversion of the elementary wave matrix can be done separately for each of the projection angles, as it follows from Eq. (15). This makes it computationally efficient, and many waves can be used in the expansion without any drastic influence in the computational cost.

### 3. NUMERICAL RESULTS

A numerical study was carried out to examine the elementary wave method proposed in this paper. A simple monopole source was used to simulate the measurements from which the Radon transform is obtained (as described in section 2.A). The sound pressure radiated by the monopole was scanned over an aperture of 95 cm, over 37 positions with an inter-spacing of approximately 2.5 cm between parallel lines. The number of projection angles was also 37, thus with a resolution of about 5°. The measurement plane was \( z_h = 5 \) cm and the reconstruction one \( z_s = 3 \) cm. Tikhonov regularization was used for the reconstruction and the regularization parameter was chosen with the generalized cross-validation method.\(^{13}\) Measurement noise of 35 dB signal-to-noise ratio (SNR) was added to the simulated measurements.

Figure 3 shows the simulated measurement of the Radon transform of the sound pressure radiated by the monopole. It is apparent that the transform is identical for every projection angle, since the sound field is rotationally symmetrical, thus the monopole is always at \( x' = 0 \).

![Fig. 3 – Radon transform of the sound field radiated by a monopole; Measurement plane \( z_h = 5 \) cm, 2000 Hz.](image)

The reconstructed sound pressure field is shown in Fig. 4. It should be noted that the values outside a radius of 47.5 cm are just padded values, because they are out of the reconstruction area. Figure 4 (right)
shows the reconstructed pressure compared to the theoretical true pressure radiated by the monopole along a straight line in the x axis. The method recovers the sound pressure satisfactorily, with an overall error below 10% (spatially averaged over the aperture). It is of particular interest to verify that the reconstruction towards the end of the aperture is accurate, tending to follow the value of the true pressure, without a drastic influence of the finite aperture. Figure 5 compares the reconstruction of the sound pressure from the elementary wave model proposed in this paper with the method based on FFT processing proposed in Ref. 8. As expected, the reconstruction based on the elementary wave model is more accurate, in particular towards the edges of the aperture where the FFT processing is notably affected by the errors related to the finite aperture. The spatially averaged error is of 28% for the FFT method, whereas of just 10% for the method proposed in this paper.

4. EXPERIMENTAL RESULTS

A simple experimental study was conducted to test the proposed method. The measurements were carried out in the anechoic chamber at the Technical University of Denmark, DTU. The sound pressure radiated by a closed box loudspeaker was scanned using a Polytec laser Doppler vibrometer and an automatic scanning robot. The loudspeaker was displaced 10 cm from the center of the aperture, at \((x, y, z) = (0, 0.1, 0)\) m, and was measured over an aperture of 90 cm with 37 parallel lines, over 37 projection angles. The
The measurement plane was \( z_h = 3 \, \text{cm} \) and the reconstruction took place at \( z_s = 2 \, \text{cm} \). A pure tone of 2400 Hz was used for the measurements, that were synchronized and performed sequentially over each position. See Fig. 6 (only one loudspeaker was excited).

**Fig. 6 – Measurement set-up. Only the loudspeaker to the left side of the picture was driven.**

The obtained results are shown in Fig. 7. The left of the figure shows the measured Radon transform, where the fact that the loudspeaker is displaced from the center of the aperture can be noticed from the curvature in the magnitude of the plot. This is because depending on the projection angle, the loudspeaker appeared between \( x' = 0 \) and \( x' = 10 \, \text{cm} \).

Figure 7 (right) shows the reconstructed sound pressure, where the position of the loudspeaker can be clearly noticed, coinciding with the maximum pressure levels. Again, the values beyond a radius of 0.45 cm are outside of the aperture, and have just been padded to a constant value in the figure, although they do not belong to the actual reconstruction. The reconstructed pressure behaves as expected, with maximum pressure about \((0, 0.1, 0) \, \text{m}\), and decaying away towards the edges of the aperture. There is some effects of scattering due to the set-up and the presence of an adjacent loudspeaker box. All in all, the results prove that the method is successful at recovering the measured sound field, and that the measured Radon transform can be used to directly reconstruct the pressure in the prediction plane satisfactorily.

**Fig. 7 – Measured Radon transform of the sound field radiated by the loudspeaker at \((x, y, z) = (0, 0.1, 0)\) m (left); Reconstructed sound pressure (right). Measurement plane \( z_h = 3 \, \text{cm} \), reconstruction plane \( z_s = 2 \, \text{cm} \), 2400 Hz.**

### 5. DISCUSSION AND FUTURE WORK

The results from this study show that the proposed method can accurately reconstruct the measured sound field, even close to the edges of the aperture, where truncation could potentially be a problem. It can be considered as a ‘patch method’, meaning that an accurate reconstruction can be achieved even if the measurement aperture is smaller than the extent of the source or the sound field radiated by it. Contrarily, conventional DFT or FFT-based methods suffer significantly from the errors associated to the finite measurement aperture, namely the replicated apertures that give rise to wrap-around error, and finite aperture...
errors. When using conventional FFT processing, zero-padding and windowing can be used to mitigate these sources of error, but at the expense of discarding useful data. It could be possible to extend artificially the measured field outside from the aperture for a better accuracy, as done in Ref. 14. This is however at the cost of increased computational complexity.

The elementary waves used in the present study are planar, which seems like the natural choice given that the measurement procedure relies on scanning the field over straight lines with a laser beam. Other approaches that make it possible to reconstruct the field over an arbitrary surface will be examined. It should be mentioned however, that due to the measurement over straight beams, conformal measurements could be challenging, and may imply large propagation distances to reconstruct the field on an arbitrary surface. This is currently ongoing work. Additionally, the reconstruction of sound fields from more realistic sources will be examined in the future, since the sound fields examined in this study are significantly simpler than the ones typically found in sound source identification problems.

6. CONCLUSIONS

In this study, a method to attain a holographic reconstruction based on the Radon transform of an acoustic field measured with the acousto-optic effect has been proposed. The method is based on an elementary plane wave expansion that makes it possible to project the Radon transform from one plane to another and directly into space domain. The method solves the problem in a least squares sense (or least norm), and thus some of the limitations related to the Fourier transform are avoided (mostly wraparound and finite aperture errors). The relation between the Radon and the wavenumber domains has been examined, and the ideal continuous case explained. A discrete implementation of the method has been introduced, from which it is possible to directly account for the propagation of the field from one plane to another and from the Radon into space domain. Both the numerical and experimental investigations show promising results that verify the enhanced accuracy of the proposed holographic reconstruction.

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