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Explicit fourth-order stiffness representation in non-linear dynamics

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Summary. In momentum-based time integration methods, the internal forces appear naturally as an approximate representation of the time integral of the internal forces over the integration interval. It is highly desirable that this force integral also represents the increment of the internal energy. A simple global form of the effective internal force is presented, in which it is represented by its algebraic mean value plus a higher order term in the form of the product of the increment of the tangent stiffness matrix at the interval end-points and the corresponding displacement increment. This explicit representation is of fourth order, and leads to the exact energy increment for systems with quartic internal energy function.

Key words: time integration, energy conservation, non-linear dynamics

Introduction

Traditionally, time integration algorithms in structural dynamics have been based on increments over the time integration interval $h = \Delta t$ and various weighted mean values, typically in the two-parameter format developed by Newmark [1], or in the extended α-modifications of the basic format, see e.g. [2]. These algorithms were initially formulated for linear problems, and their properties characterized by spectral analysis. In the case of non-linear problems the basic collocation format, in which the state variable increments are expressed in terms of averages of forces corresponding to specific times t_n, suffers from the basic weakness that the corresponding energy increment over the time interval can not be expressed in this format. Thus, the collocation format intrinsically breaks with the property of energy conservation for non-linear problems.

About two decades ago an alternative approach to time integration algorithms in dynamics was developed based on representing the time integral of the state-space equations of motion. The original development by Simo and co-workers [3] was based on the observation that in linear elasticity with Green strains the effective internal force over a time increment can be represented via the product of the mean value of the strain gradient and the mean value of the stress. The procedure was later generalized to non-linear elasticity by Gonzalez [4], using a stiffness representation similar to that of the classic BFGS method. These methods provide momentum and energy balance within a second-order time accurate scheme. However, the format has the drawback that the internal force is formed in terms of a product of averages at the element level, and therefore requires reformulation of traditional element procedures, in which the internal force is formed for a particular state associated with the discrete points t_n used in the time integration algorithm. It was later shown by Krenk [5, 6] that in the case of linear elasticity with Green strains energy conservation could be attained by supplementing the mean value of the internal forces with an additional stiffness term formed by the increment of the geometric stiffness matrix. This term can be formed at the global level, and thus permits inclusion as a simple modification of existing finite element codes.

The present paper develops a fourth-order representation of the internal force in terms of values of the internal force and the tangent stiffness matrix at the integration interval end-points at t_n and t_{n+1}. The result is based on Taylor series expansions, and therefore does not require any special format of the internal energy. For energy functions that are up to quartic in the displacement components the energy is conserved exactly, and otherwise to the fourth order.

Momentum based state-space equations

Consider a mechanical system, described in terms of displacements contained in the column vector $\mathbf{u}$. The internal forces are assumed to be given in terms of the potential $G(\mathbf{u})$ in the form

$$\mathbf{g}(\mathbf{u}) = \boldsymbol{\nabla}_{\mathbf{u}} G(\mathbf{u}) \tag{1}$$

where $\boldsymbol{\nabla}_{\mathbf{u}}$ denotes the derivatives with respect to the displacement components of $\mathbf{u}$. For convenience the resulting components are defined to be in column format. The mass matrix $\mathbf{M}$ is assumed to be constant, as e.g. for models using isoparametric elements. The corresponding equation of motion is

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{g}(\mathbf{u}) \;=\; \mathbf{f}(t) \tag{2}$$

where $\mathbf{f}(t)$ is the load vector, and $\mathbf{C}$ is a matrix representing linear viscous damping.

It is convenient to introduce the velocity $\mathbf{v} = \dot{\mathbf{u}}$ and work in terms of the state-space variables $[\,\mathbf{u}^T, \mathbf{v}^T\,]$. The equation of motion can then be expressed by the following symmetric set of first-order differential equations,

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}(\mathbf{u}) \\ -\mathbf{M}\,\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{3}$$

The key step in momentum based integration methods is the use of a time integrated form of these equations, here expressed as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} + h \begin{bmatrix} \mathbf{g}_* \\ -\mathbf{M}\,\bar{\mathbf{v}} \end{bmatrix} = h \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

where $\mathbf{g}_* \simeq h^{-1} \int \mathbf{g}\, dt$ and the time mean values of $\mathbf{v}$ and $\mathbf{f}$ are represented by their algebraic mean.

The important point now is to identify a suitable definition of the representative mean value $\mathbf{g}_*$, such that it also satisfies the energy increment relation

$$\Delta G \;=\; \Delta\mathbf{u}^T \boldsymbol{\nabla}_{\mathbf{u}} G_* \;=\; \Delta\mathbf{u}^T \mathbf{g}_* \tag{5}$$

In [3] and [4] this problem was solved for a linear/non-linear elastic body via properties in the mean state at $t_{n+1/2}$, while [5] obtained an explicit expression in terms of the increment of the geometric stiffness for a linear elastic body with Green strains. In the following section a general expression for $\mathbf{g}_*$ of order four is obtained for a general potential function $G(\mathbf{u})$ in terms of the increment of the full stiffness matrix.

Representation of the internal force

The present representation of $\mathbf{g}_*$ is obtained from a suitably modified form of the Taylor expansion of the potential $G(\mathbf{u})$. For this purpose the displacement increment is parameterized over the time interval $[t_n, t_{n+1}]$ as

$$\mathbf{u} \;=\; \mathbf{u}_{n+1/2} + \xi \Delta\mathbf{u} \tag{6}$$

This gives the following expression for the energy increment

$$\Delta G = \int_n^{n+1} \mathbf{g}(\mathbf{u})^T d\mathbf{u} = \Delta\mathbf{u}^T \int_{-1/2}^{1/2} \mathbf{g}(\mathbf{u}_{n+1/2} + \xi\Delta\mathbf{u})\, d\xi \tag{7}$$

The internal force is now expanded in terms of ξ as

$$\mathbf{g}(\mathbf{u}) = \mathbf{g}(\mathbf{u}_{n+1/2}) + \xi\mathbf{g}'(\mathbf{u}_{n+1/2}) + \tfrac{1}{2}\xi^2\mathbf{g}''(\mathbf{u}_{n+1/2}) + \cdots \tag{8}$$

When substituting this expansion into the integral in (7) it is seen that only the even powers of ξ contribute, and when including only the first two contributing terms

$$\Delta G = \Delta\mathbf{u}^T \Big[\mathbf{g}(\mathbf{u}_{n+1/2}) + \tfrac{1}{24}\mathbf{g}''(\mathbf{u}_{n+1/2}) + \mathrm{O}(h^4) \Big] \tag{9}$$

where it has been used that $\Delta\mathbf{u}$ is of order h, and that each derivative with respect to ξ introduces a factor $\Delta\mathbf{u}$ of order h. The terms inside the square brackets define the effective internal force as

$$\mathbf{g}_* = \mathbf{g}(\mathbf{u}_{n+1/2}) + \tfrac{1}{24}\mathbf{g}''(\mathbf{u}_{n+1/2}) + \mathrm{O}(h^4) \tag{10}$$

The first term in this representation is evaluated in the mean displacement state $\mathbf{u}_{n+1/2}$. This would be a serious drawback, and this term is therefore reformulated by use of the Taylor expansion (8) to express the algebraic mean of the internal force,

$$\tfrac{1}{2}[\,\mathbf{g}_{n+1} + \mathbf{g}_n\,] = \mathbf{g}(\mathbf{u}_{n+1/2}) + \tfrac{1}{8}\mathbf{g}''(\mathbf{u}_{n+1/2}) + \mathrm{O}(h^4) \tag{11}$$

When this relation is used to eliminate $\mathbf{g}_{n+1/2}$ in (10) the effective force is expressed as

$$\mathbf{g}_* = \tfrac{1}{2}[\,\mathbf{g}_{n+1} + \mathbf{g}_n\,] - \tfrac{1}{12}\mathbf{g}''(\mathbf{u}_{n+1/2}) + \mathrm{O}(h^4) \tag{12}$$

The final task now is to obtain a representation of the second term without explicit reference to the mean state $\mathbf{u}_{n+1/2}$.

The second term is expressed in terms of the increment of the tangent stiffness, $\Delta\mathbf{K}$. In order to obtain this expression the first derivative of the internal force with respect to the non-dimensional parameter ξ is expressed as

$$\mathbf{g}'(\mathbf{u}) = \frac{d\mathbf{g}}{d\mathbf{u}}\Delta\mathbf{u} = \mathbf{K}(\mathbf{u})\Delta\mathbf{u} \tag{13}$$

In the present formulation the displacement increment $\Delta\mathbf{u}$ is fixed, and the second derivative of the internal force with respect to the non-dimensional parameter ξ therefore follows as

$$\mathbf{g}''(\mathbf{u}) = \frac{d\mathbf{K}}{d\xi}\Delta\mathbf{u} \simeq \frac{\Delta\mathbf{K}}{\Delta\xi}\Delta\mathbf{u} = \Delta\mathbf{K}\Delta\mathbf{u} \tag{14}$$

Substitution of this representation into the expansion (12) then gives the final form

$$\mathbf{g}_* = \tfrac{1}{2}\big[\,\mathbf{g}_{n+1} + \mathbf{g}_n\,\big] - \tfrac{1}{12}\Delta\mathbf{K}\,\Delta\mathbf{u} + \mathrm{O}(h^4) \tag{15}$$

In this form the effective internal force is expressed to fourth order entirely by the internal force and the stiffness matrix at the integration interval end-points. The form of this representation is similar to that obtained in [5] for the special case of linear easticity in terms of the Green strain, where the second term was given in terms of the increment of the geometric stiffness as $-\frac{1}{4}\Delta\mathbf{K}_g\,\Delta\mathbf{u}$. However, the present result dos not require any special form of the energy function, and no separation of the stiffness matrix is involved.

Fourth-order conservation algorithm

The discretized form (4) of the equations of motion is now expressed by use of the fourth-order representation (15) of the internal force, whereby

$$\begin{bmatrix} \mathbf{C} - \frac{1}{12}h\Delta\mathbf{K} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} + \frac{h}{2} \begin{bmatrix} \mathbf{g}(\mathbf{u}_{n+1}) + \mathbf{g}(\mathbf{u}_n) \\ -\mathbf{M}\,\mathbf{v}_{n+1} - \mathbf{M}\,\mathbf{v}_n \end{bmatrix} = h \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \tag{16}$$

The viscous damping in terms of $\mathbf{C}$ can be replaced or supplemented by an algorithmic damping by inserting additional terms in the diagonal of the first block matrix, following the procedure in [6].

In the solution procedure the the velocity $\mathbf{v}_{n+1}$ is eliminated in the first equation by use of the second, and a non-linear equation is obtained in $\mathbf{u}_{n+1}$. Subsequently, the velocity $\mathbf{v}_{n+1}$ is obtained from the second equation, using the now known value of $\mathbf{u}_{n+1}$.

The present conservative time integration procedure with fourth-order stiffness representation is illustrated by some simple examples, demonstrating the improved time representation in the solution, as well as the use in problems with internal energy representation that is not directly of fourth degree in the displacements.

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