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Okulov, Valery; van Kuik, G.A.M.

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The Betz–Joukowsky limit: on the contribution to rotor aerodynamics by the British, German and Russian scientific schools

Valery L. Okulov

Department of Mechanical Engineering and Center for Fluid Dynamics, Technical University of Denmark, DK-2800 Lyngby, Denmark; Kutateladze Institute of Thermophysics, Siberian Branch, Russian Academy of Sciences, pr. Akademika Lavrent'eva 1, Novosibirsk, 630090 Russia

E-mail: vaok@mek.dtu.dk

Gijs A.M. van Kuik,

Delft University Wind Energy Research Institute, Faculty of Aerospace Engineering, Technical University of Delft, Kluyverweg 1, 2629 HS Delft, The Netherlands

E-mail: G.A.M.vanKuik@tudelft.nl

Abstract

The derivation of the efficiency of an ideal wind turbine has been attributed to famous scientists of the three aerodynamic research schools in Europe during the first decades of the previous century: Lanchester, Betz and Joukowsky. However, detailed reading of their classical papers has shown that Lanchester did not accept Froude's result that the velocity through the disc is the average of the velocities far up- and downstream, by which his solution is not determined. Betz and Joukowsky used vortex theory to support Froude's result, and derived the ideal efficiency of a wind turbine at the same time. This efficiency has been known as the Joukowsky limit in Russia and as the Betz limit everywhere else. Due to the contribution of both scientists, this result should be called the Betz-Joukowsky limit everywhere.

1. Introduction

The result mentioned in the title is famous: no more than 59.3% of the kinetic energy of fluid contained in a stream tube having the same cross section as the area of a rotor disc (or any energy transformer) can be converted to useful work by the disc. Certainly, this result is right only if the stream tube is free from external influences, for example, the stream tube should not be inside diffusers or pipe sections. Indeed it is impossible to stop a wind/fluid motion in the stream tube to take 100 % efficiency[1] because this implies a full blockage of the flow downstream of the disc. The flow can only be reduced, not stopped, so there is a rate of deceleration that gives maximum power conversion: 59.3 % of the kinetic energy. This was established by the actuator disc theory based on the conservation laws of the flow. In this theory, any device is replaced by a permeable disc with a distributed load yielding the same overall thrust at the real device. Despite this far-reaching abstraction this theory is the base of wind turbine aerodynamics. Sometimes this limit is named as the «Carnot cycle» of wind energy because the value does not depend on the type of energy transformer.

For a long time, this result about the maximum of kinetic energy which can be utilized by an ideal wind-driven generator, was known as the Betz limit published in 1920 [1] (Betz was a pupil of Prandtl and then headed his laboratory). In 1979 Bergey [2] has proposed that Lanchester [3] should be associated with the result too because of his 1915 publication. In 2007 van Kuik [4] found that Joukowsky [5] derived the same result in 1920. Taking into account the high importance of this result, our current study addresses again the history of the discovery. Photographs of the Joukowsky, Betz and Lanchester are shown in fig. 1

2. The actuator disc theory: Retrospect

The actuator disc (or momentum; or slip-stream) theory is the oldest mathematical representation of a screw, propeller or wind turbine etc in fluid dynamic calculations. The load on a real rotor is replaced by a pressure distribution on an infinitely thin, permeable disc with the same diameter. In its most elementary presentation, this load is uniform and normal, with the disc placed in an axial flow perpendicular to the flow direction. This actuator disc concept is still used as an easy qualitative diagnostic model, and any textbook on rotary wing or rotor aerodynamics starts with it. The three European aerodynamic research schools that were famous in the first half of the XXth century have contributed significantly to this theory: the British school led by Froude and Lanchester, the German school led by Prandtl and Betz, and the Russian school led by Joukowsky and Vetchinkin. These contributions are reviewed in retrospect.

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[1] The efficiency is defined as the ratio of useful work performed by the disc divided by the kinetic energy of the undisturbed flow contained in a stream tube with the same cross-section as the disc.

2.1 The British School

The idea to replace a rotor by an actuator disc goes back to the work of Rankine [6]. However only in 1889 R.E. Froude [7] has for the first time found a correct dynamic interpretation of the actuator disc action showing that for such a theoretical propeller one half the acceleration must take place before the propeller and the other half behind it. Unfortunately, the discussion on the question whether the contraction or expansion of the streamtube takes place before or behind the disc continued after Froude's paper, despite his formal mathematical treatment. In the Appendix to his paper, §27, Froude derives the energy balance (the work done by the disc is thrust times velocity through the disc, which equals the change in kinetic energy of the flow in the streamtube times the mass flow in the tube) and the momentum balance (the thrust equals the mass flow times the change of velocity). This gives the result that the velocity at the disc is the average of the velocities far upstream and far downstream.

Professor V.P. Vetchinkin [8-9], who was a pupil of Joukowsky, sought an explanation for the denial of Froude's result in the misunderstanding of the relation between the action of a disc and of real rotor blades. Most scientists at that time thought, erroneously, that the flow before the rotor plane is undisturbed, then receives a full speed alteration when it moves through the rotor blades, after which the flow behind the rotor is undisturbed too. Another incorrect interpretation of an ideal rotor, based on Parsons' remarks to the Froude's article [10] assumes that the total flow alteration is received before and during the rotor passage only, with the flow behind the rotor moving uniform and without disturbances. Followers of Parsons' theory said that it is impossible to find any reason or force for the flow acceleration/deceleration in the free stream tube behind the rotor plane when the rotor action has ended. Though the assumption of Parsons' theory results in incompatible equations for the ideal work and energy in the wake, the theory was very popular among famous English scientists in beginning of XX century. From 1910 to 1915 there was a lively discussion on both Froude's and Parsons' theories between professors Henderson, Froude, Parsons and Lanchester in pages of the popular English edition: «*Transactions of the Institution of Naval Architects*» (see issues of 52, 53, 55 and 56). An author of the last article is F.W. Lanchester [3]. In the introduction to this article he wrote: - "The present investigation takes for its starting point the simplified or idealized conception of propeller due to Mr. R.E. Froude …" – then he remarked on the discussion mentioned above and concluded – “Without definitely entering or taking part in the dispute in question, the present contributor proposes to review the theory from its foundation, in order to make sure of his own ground…”. In his analysis, Lanchester supports the energy and momentum balance as defined by Froude, as is clear from his statements at p. 108. However, he continues the discussion on the ‘difficulties of regime’ started by Froude, in particular on the pressure discontinuity at the disc edge. Although Lanchester says that ‘*the admitted difficulty relating to the edge of the actuator is probably more apparent than real’* his next step is to substitute the continuously operating disc by an intermittendly operating disc shedding vortex rings into the flow. Lanchester states that a considerable portion of the change in kinetic energy is now to be found in the outer portions of the vortex rings, so outside the streamtube passing through the disc. According to modern insights in vorticity dynamics, Lanchester’s statement is incorrect since he should include also the pressure- and unsteady terms in the energy equation. It is here where Lanchester deviates from Froude and leaves the possibility open that velocity at the disc is not the average of the velocites far up- and downstream. In the Appendix, Lanchester’s derivation is presented, showing how close he was to a firm assessment of the maximum efficiency. In effect, Lanchester tried to find a compromise between both Froude’s and Parsons’ theories of the ideal propeller/turbine but his symbiosis of the theories was nearly to Parsons’ point of view as it was noticed by Vetchinkin. Indeed, Lanchester’s model of the ideal wind turbine, described in Problem II of [3] and partly repeated in the Appendix, is a symbiosis of both theories with transition parameter Q ($1 < Q < 2$). When $Q = 1$ the symbiosis coincides with the Froude theory and for $Q = 2$ it is the Parsons theory (see Fig. 3 which is copy of Fig. 6 from [3]). For the symbiosis model Lanchester has derived a formula of the power coefficient which is equal to $16/27Q^2$ and tends to the maximum of the Froude’s power coefficient if $Q = 1$. Lanchester suggests that Q should be around 1.5..

Bergey’s conclusion [2] that Lanchester is co-author of the Betz limit is not valid, since it assumes that Lanchester has adopted $Q = 1$. Bergey ignores the Q discussion and presented the derivations and equations of Lanchester after substitution of $Q = 1$, whereas Lanchester himself included Q in the equations yielding results for the limit conditions $Q = 1$ and 2.

In his paper, Bergey discusses the fact that the Prandtl school did not refer to Lanchester’s work, without finding a satisfactory answer. Indeed Lanchester was well-known in Germany and in Russia. His article [3] was well known too. It has been published in a well accessible edition, and it was referred to by Joukowsky in his famous work "Vortex theory of screw propeller" [11-14] (in the last article of this cycle [14]) and in Vetchinkin [9]. The Russian and German schools knew each other very well and cooperated, e.g. in a common responsibility for a scientific journal, see fig. 2. The absence of the reference in Prandtl's school was because they could not support it.

As a preliminary conclusion, Lanchester's name should not be linked to the Betz limit, which results directly from Froude's theory. It is noteworthy to realize that he and his biographies never claimed the rights to this result.

2.2 The German and Russian School

In the meantime, more and more scientists in the world became supporters of the Froude's theory as Vetchinkin noted it in [9]. In accordance with the Vetchinkin's remarks Bendemann [15] wrote in 1910 that professor Finsterwalder has proved in his 1904 lectures that the induced velocity in the far wake behind a rotor becomes double its value in the rotor plane. He drew the same conclusion as Froude but it is not clear whether he did know his article of 1889. In 1912, during the second Russian aeronautic congress, Sabinin reported this fact too (see the record in sixth footnote of [14]) and Vetchinkin reproduced Finsterwalder's proof in [8]. In 1917 Bothezat has generalized the result about the doubling of the induced velocity in the wake for actuator discs producing not only a forward but also a rotary movement in the wake [16]. Finally Joukowsky formulated in 1918 the modern state of the Froude's theory in §6 of [14]. This history has been supported and extended by Hoff [17] who indicated Finsterwalder as the scientist who established the theory too, which was extended afterwards by Bendemann in [15] and completed by Prandtl in an appendix to Betz's paper [16]. Probably this list of supporters of Froude's theory may be expanded and each of the scientists could have derived the limiting value of the Froude's theory in an application to wind turbine. Indeed the value consists in Lanchester theory [3] as a limit at $Q = 1$. Sabinin [18] mentions that Vetchinkin was the first, in 1914, who expanded Froude's theory of propellers to wind turbines but this does not become clear from his publications.

Although Froude's theory was accepted by many scientists, it was not yet possible to show a connection between the abstraction of the actuator disc and the action of real blades on the flow. During the first two decades of the XX century this led to a struggle of viewpoints, that was resolved by vortex theory.

3. The vortex theory of propellers

The first article of Professor N.E. Joukowsky from his cycle «Vortex theory of screw propeller» [11] has been published in 1912. In this article, he created the vortex model of a propeller based on a rotating horseshoe vortex, which expanded the first elementary vortex model of a wing with a finite span. In his vortex theory each of the blades is replaced by a lifting line about which the circulation associated with the bound vorticity is constant, resulting in a free vortex system consisting of helical vortices with finite cores trailing from tips of the blades and a rectilinear hub vortex, as sketched in Fig. 4. Vetchinkin commented in [9] that Joukowsky's vortex theory has finally confirmed Froude's actuator disc theory. In 1918 in the last, fourth, article of the cycle [14] Joukowsky expounds the theory of an ideal propeller based on the Froude's actuator disc theory in full and with the dimensionless view accepted today.

Simultaneously in Germany A. Betz worked on the creation of the propeller vortex theory. About the actuator disc theory, he remarked in [19]: "the point, in which the old propeller slip-stream theory (*the Froude's theory* – ***rem. by authors***) needed to be supplemented, was the assumption that the thrust could be distributed at will over the surface of the propeller disc… It may, however be here noted that the difference in comparison with the uniform distribution is not so great as appears at the first glance." Then he referred to Föttinger's propeller horseshoe vortex model [20] consisting of infinitely thin vortex lines, which confirmed the main findings of Joukowsky's model. As the next step Betz proposed a new model in 1919 [21]. In Betz's model of the vortex theory each of the rotor blades is replaced by a lifting line of which the circulation is associated with bound vorticity, with a free vortex sheet being shed continuously from the trailing edge of the rotating blade and moving with constant velocity in axial direction (Fig. 5). In contrast with the horseshoe vortex model this alternative propeller theory used Prandtl's vortex model of a wing with an elliptic distribution of the loading along the span [22-23]. Thus, a field of well-defined vortices is connected with the distribution of the propeller thrust, or the lift of a rotating wing. Betz concluded in [19] that the motion of the fluid is definitely determined by the vortices existing in it, and that the flow due to the thrust distribution may be calculated by means of this concept of vortices.

The conceptual ideas of Joukowsky's and Betz's interpretations of the propeller vortex theory have allowed proving for Froude's actuator disc theory as a unique and correct elementary theory for propellers. In 1913-1918 Joukowsky [11-14] and in 1919 Betz [21] have shown a connection between an abstraction of the actuator disc and the real blades action on the flow for the first time.

4. The Betz-Joukowsky limit

In 1920, famous aerodynamists (Joukowsky and Betz) have independently published articles to develop Froude's theory to the theory of the ideal wind turbine in which the value of the maximal ideal work which can be extracted from the kinetic energy of wind was found [1, 5]. Munk [24] did the same, also in 1920. In addition to this Hoff remarked that his article with the same topic [17] was written somewhat later than articles [1] and [24] but was independent of them too. Maybe there are more unknown works. It was a natural result after the confirmation of Froude's actuator disc theory by vortex theory in [11-14] and [21] (see table 1) because the development of the actuator disc theory to wind turbines looks simple and ordinary. Nevertheless, we select here only two names because the two independent publications by Joukowsky [5] and Betz [1] in 1920 are the result of their great achievements in [11-14] and [21]. Their contemporaries made the same choice too to pay attention to only these scientists for their confirmation of Froude's theory. Indeed, everywhere in the world the ideal efficiency of a wind turbine is known as the Betz limit or Betz law without a reference to somebody else. In Russia this result is known as the Joukowsky limit, with the reference that it is known as the Betz limit outside Russia. Below we try to find answer for this difference.

The paper of Betz [1] has a title that shows the topic clearly: the maximum efficiency of a wind turbine. Joukowsky's article [5] has a quite special purpose. It was a response to inventor Vinogradov, who promised to create a wind turbine with unprecedented efficiency. Joukowsky answered him by a conclusion about the maximum of wind energy utilization for the ideal wind turbine described in the first paragraph of the foregoing article. In the same article, he has expanded the theory for case of the ideal wind turbine with rotation in wake behind a rotor and with an additional assumption about constancy of the circulation. In addition to this Betz has published in 1926 the remarkable book "The Wind Energy and its utilization by windmills" which made the name of Betz well-known name amongst wind energy engineers. Furthermore, the paper of Betz has been published in a journal, whereas, the paper of Joukowsky was part of the Transactions of his scientific institute, with a possibly more limited distribution. Joukowsky was 73 in 1920 and it was the last article in his life. The followed illnesses and death have given up an opportunity to continue his works on the topic. These are reasons why his work went unnoticed for the scientific community in the world. Moreover, in 1927 the attention of compatriots has been drawn away by the erroneous theory of an ideal propeller of Sabinin [18]. Unfortunately, his wrong theory finds now followers again since it used in some Russian textbooks and manuals. Sabinin wished to create a distinct theory from Froude's theory, for the operating regime with a turbulent wake where the model of ideal wind turbine is not valid. At first glance, his theory was constructed in strong accordance with vortex theory unlike the Parsons and Lanchester case with $Q > 1$. Nevertheless, he used a wrong idea to calculate the lift forces by a starting-vortex that travels downstream and keeps the shape of a vortex solenoid. The last assumption is wrong because it neglects a roll-up mechanism of the starting vortex, by which the real flow and pressure are totally different from Sabinin's model. In his theory, the maximum efficiency becomes a little larger (68.7%) than the Betz-Joukowsky value. If we write the Sabinin theory in terms of the Lanchester theory it is easy to find that $Q = 1.137$, and we meet the same contradiction that was in Parsons and Lanchester theories.

5. Conclusions

Although Lanchester was close to calculate the limit efficiency of a wind turbine, he did not do so since he did not include Froude's result that the velocity through the disc is the average of the velocities far up- and downstream. .

This limit has been known as Betz's limit everywhere in the world and as Joukowsky's limit in Russia only. Due to the contribution of both scientists, this result should be identically called the Betz-Joukowsky limit everywhere - in Russia and in the world too.

Acknowledgements

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Appendix

Fragment of the article "A contribution to the theory of propulsion and the screw propeller" by F.W. Lanchester [3], PROBLEM II. – THE WINDMILL

E_t = energy per second in windstream engaged.

m_t = mass per second of air engaged.

V = velocity of wind (a constant).

v = residual velocity.

u = velocity when passing through actuator.

w = force on actuator.

$$Q = \frac{\text{wind energy}}{\text{work done}} \quad \text{or} \quad = \frac{E_t}{\text{power developed}}$$

(*Lanchester has introduced the parameter Q as a measure of the incompatibility for the ideal work and energy in the wake –* ***rem. by authors***)

$$E_t = m_t \frac{V^2 + v^2}{2} \tag{1}$$

$$w\,u = m_t \frac{V^2 - v^2}{2Q} \tag{2}$$

and momentum/sec. –

$$w = m_t (V - v) \tag{3}$$

or by (2) and (3) –

$$u = \frac{V^2 - v^2}{2Q(V - v)} = \frac{V + v}{2Q} \tag{4}$$

If p = work done per unit mass/sec. (per unit m_t) –

$$p = \frac{w u}{m_t} \text{ or by (2)} = \frac{V^2 - v^2}{2Q} \tag{5}$$

By (4) –

$$\frac{du}{dv} = \frac{1}{2Q} \tag{6}$$

By (5) –

$$\frac{dp}{dv} = -\frac{v}{Q} \tag{7}$$

By (6) and (7) –

$$\frac{dp}{du} = -\frac{2Q\,v}{Q} = -2v \tag{8}$$

And when *pu* is maximum –

$$\frac{dp}{du} = -\frac{p}{u}$$

or by (4), (5) and (8) –

$$\frac{V^2 + v^2}{2Q} \cdot \frac{2Q}{V + v} = 2v$$

or –

$$V - v = 2v$$

$$V = 3v \qquad (9)$$

Thus the maximum work is got out of the wind when its residuary velocity is one-third of initial velocity, and this ratio is independent of the value of *Q*.

Assuming best relation $v = \frac{V}{3}$. By (4)

$$u = \frac{V + v}{2Q} = \frac{4}{3}\frac{V}{2Q} = \frac{2}{3}\frac{V}{Q} \qquad (10)$$

The limiting conditions are $Q = 1 \quad u = \frac{2}{3}V$ and $Q = 2 \quad u = \frac{1}{3}V$. The first corresponds to the Froude condition, the second is analogous to (but in sense of reverse of) the condition attributed to Parsons, thus (Fig. 3).

We will now find an expression to represent the *available power*, expressing this in terms of a standard represented by the energy of the free passing per second across an aria equal to that of the actuator. We will denote the available power so expressed by the symbol ζ.

Assuming best condition, *i.e.*, $V = 3v$. By (2)

$$\text{Power developed} = w\,u = \frac{4m_t V^2}{9Q} \qquad (11)$$

where, by (10) $m_t = Au\,\rho = \frac{2AV\rho}{3Q}$. (11) becomes

$$\frac{8AV^3\rho}{27Q^2} \qquad (12)$$

Power represented by "free wind" on area *A*

$$\frac{AV^2\rho}{2} \qquad (13)$$

or

$$\zeta = \frac{16}{27Q^2}$$

For the limiting values –

$$Q = 1, \qquad \zeta = \frac{16}{27} = 0.6 \text{ approximately}$$

$$Q = 2, \qquad \zeta = \frac{4}{27} = 0.15$$

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N.E. Joukowsky A. Betz F.W. Lanchester

Fig. 1: Three pioneers in actuator disc theory[2].

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Fig. 2: Title page of a scientific journal, showing the connection between the Russian and German aerodynamic schools

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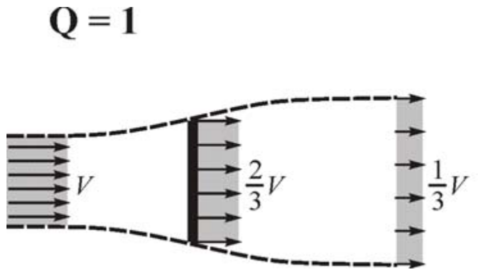
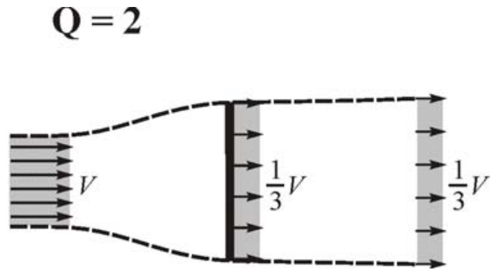
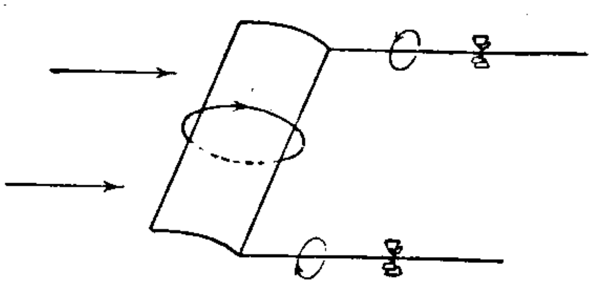


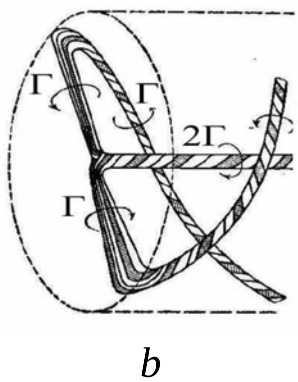


a *b*

Fig.3 (a) Correct model of the Froude's actuator disc;
(b) Erroneous model by Parsons.



a



b

Fig.4 (*a*) the first elementary vortex model of a wing with a finite span based on a single horse-shoe vortex reported by Prandtl at the Gottingen congress in 1913; (*b*) Joukowski's vortex model of a propeller based on a rotating horse-shoe vortex in the accordance with Prandtl' model for a wing [7].

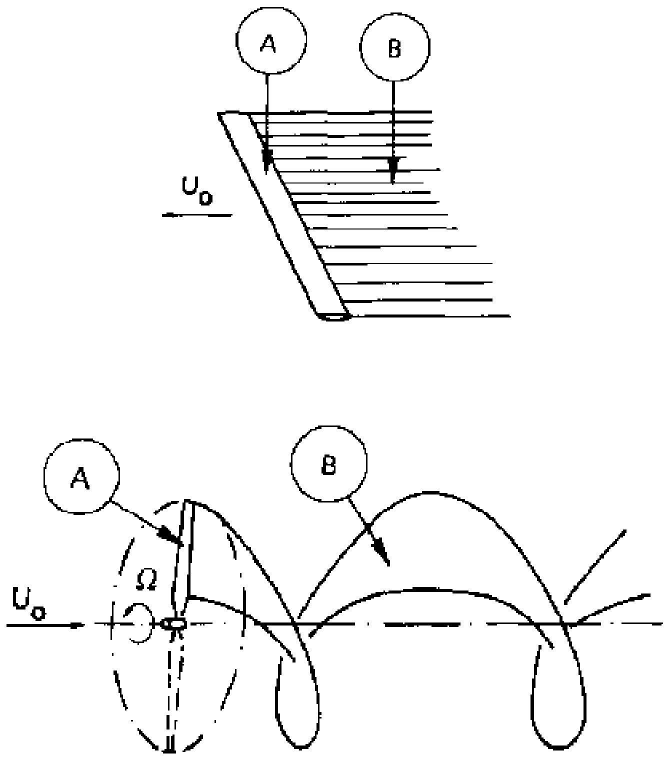


Fig.5 (a) Refined Prandtl's vortex model of a wing with elliptic load distribution published in 1918 [13];
(b) Betz's vortex model of a propeller (1919) based on the new Prandtl's solution [12].

Table 1. History of the Betz limit: the main contributions and contributors

Contributions	Year and scientist of		
	British school	German school	Russian school
Actuator disc theory	1865 W. Rankine		
	1889 R.E. Froude		
		1904 Finsterwalder	
		1910 Bendemann	
			1912 Sabinin
			1913 Vetchinkin
			1917 Bothezat
			1918 Joukowsky
		1919 Prandtl	
Development of the actuator disc theory based on wrong approaches by Parsons etc.	*1911 Parsons*		
	1915 Lanchester		
			1927 Sabinin
Corroboration of the actuator disc theory by the vortex theory			1913 Joukowsky
		1919 Betz-	
Expansion of the actuator disc theory to wind turbines and formulation of the limit for the power coefficient			***February*, 1920 Joukowsky**
		August, 1920 Munk	
		***September*, 1920 Betz**	
		July, 1921 Hoff	