Finite groups in which some particular subgroups are TI-subgroups

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FINITE GROUPS IN WHICH SOME PARTICULAR SUBGROUPS ARE TI-SUBGROUPS

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Abstract. We prove that $G$ is a group in which all noncyclic subgroups are TI-subgroups if and only if all noncyclic subgroups of $G$ are normal in $G$. Moreover, we classify groups in which all subgroups of even order are TI-subgroups.

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1. INTRODUCTION

All groups in this paper are considered to be finite. Let $G$ be a group and $M$ a subgroup of $G$. It is known that $M$ is called a TI-subgroup of $G$ if $M \cap M^g = 1$ or $M$ for any $g \in G$.

In [5], Walls studied groups in which all subgroups are TI-subgroups. In [1], Guo, Li and Flavell classified groups in which all abelian subgroups are TI-subgroups. As a generalization of [1], we showed in [3] that $G$ is a group in which all nonabelian subgroups are TI-subgroups if and only if all nonabelian subgroups of $G$ are normal in $G$, and we showed in [4] that $G$ is a group in which all nonnilpotent subgroups are TI-subgroups if and only if all nonnilpotent subgroups of $G$ are normal in $G$.

In this paper, as a further generalization of above references, we first classify groups in which all noncyclic subgroups are TI-subgroups. We call such groups NCTI-groups. For NCTI-groups, we have the following result, the proof of which is given in Section 2.

**Theorem 1.** A group $G$ is an NCTI-group if and only if all noncyclic subgroups of $G$ are normal in $G$.

**Remark 1.** It is easy to show that any NCTI-group is solvable but it might not be supersolvable. For example, the alternating group $A_4$ is an NCTI-group but $A_4$ is nonsupersolvable.
We say that a group $G$ has nontrivial even order if $|G|$ is even but $|G| \neq 2$. In the following, we will classify groups in which all subgroups of even order are TI-subgroups. We call such groups EOTI-groups. For EOTI-groups, we have the following result, the proof of which is given in Section 3.

**Theorem 2.** A group $G$ is an EOTI-group if and only if one of the following statements holds:

1. all subgroups of $G$ of nontrivial even order are normal in $G$.

2. $G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel $Z_p$ and complement $\langle g \rangle$, where $p$ is an odd prime and $o(g) = 2n$ for some $n > 1$.

**Remark 2.** We can easily get that any EOTI-group is solvable. But the alternating group $A_4$ shows that a EOTI-group might not be supersolvable.

Arguing as in proof of Theorem 2, we can obtain the following three results, which are generalizations of Theorem 1, [3] and [4] respectively. Here we omit their proofs.

**Theorem 3.** A group $G$ is a group in which all noncyclic subgroups of even order are TI-subgroups if and only if all noncyclic subgroups of $G$ of even order are normal in $G$.

**Theorem 4.** A group $G$ is a group in which all nonabelian subgroups of even order are TI-subgroups if and only if all nonabelian subgroups of $G$ of even order are normal in $G$.

**Theorem 5.** A group $G$ is a group in which all nonnilpotent subgroups of even order are TI-subgroups if and only if all nonnilpotent subgroups of $G$ of even order are normal in $G$.

2. **Proof of Theorem 1**

**Proof.** The sufficiency part is evident, we only need to prove the necessity part.

Let $G$ be an NCTI-group. Assume that $G$ has at least one nonnormal noncyclic subgroup. Choose $M$ as a nonnormal noncyclic subgroup of $G$ of largest order. Then, for any noncyclic subgroup $K$ of $G$, we have that $K$ is normal in $G$ if $M < K$.

We claim that $M = N_G(M)$. Otherwise, assume that $M < N_G(M)$. Let $L$ be a subgroup with $M < L \leq N_G(M)$ such that $M$ has prime index in $L$. By the choice of $M$, we have that $L$ is normal in $G$. Since $M$ is not normal in $G$, $N_G(M) < G$. Take $h \in G \setminus N_G(M)$. We have $M^h < L^h = L$. Since $M$ is a normal maximal subgroup of $L$ and $M^h \neq M$, we have $L = MM^h$. Note that $M$ is a TI-subgroup of $G$, which implies that $M \cap M^h = 1$. Then $|L| = |MM^h| = |M||M^h|$. It follows that $|M| = |M^h| = \frac{|L|}{|M|} = |L : M|$ is a prime, this contradicts that $M$ is noncyclic. Thus $M = N_G(M)$.

Then, by the hypothesis, $M \cap M^a = 1$ for every $a \in G \setminus N_G(M) = G \setminus M$. We have that $G$ is a Frobenius group with $M$ being a Frobenius complement. Since $G$
is an NCTI-group, it follows that all nonabelian subgroups of $G$ are TI-groups. By [3], we have that all nonabelian subgroups of $G$ are normal in $G$. Then $M$ is abelian. It follows that every Sylow subgroup of $M$ must be cyclic by [2, Theorem 10.5.6]. This implies that $M$ is cyclic, a contradiction. Thus $G$ has no nonnormal noncyclic subgroup.

\[ \square \]

3. Proof of Theorem 2

Proof. (1) We first prove the necessity part. Let $G$ be a EOTI-group. Assume that $G$ has at least one nonnormal subgroup of nontrivial even order. Let $Q$ be any nonnormal subgroup of $G$ of nontrivial even order. Choose $L$ as a nonnormal subgroup of $G$ of nontrivial even order of largest order such that $Q < L$.

We claim that $L = N_G(L)$. Assume that $L < N_G(L)$. Let $K$ be a subgroup with $L < K \leq N_G(L)$ such that $L$ has prime index in $K$. By the choice of $L$, we have that $K$ is normal in $G$. Let $y$ be an element of $G \setminus N_G(L)$. Then $L \cap L^y = 1$ by the hypothesis. It follows that $K = LL^y$ since $L$ is a normal maximal subgroup of $K$. Then $|L| = |L^y| = |K : L|$ is a prime, a contradiction. Thus $L = N_G(L)$.

It follows that $G$ is a Frobenius group with complement $L$. Let $N$ be the Frobenius kernel of $G$. We have $G = N \rtimes L$.

We claim that $L$ is a maximal subgroup of $G$. If $L$ is not a maximal subgroup of $G$. Let $M$ be a maximal subgroup of $G$ such that $L < M$. It is obvious that $M$ also has nontrivial even order. By the choice of $L$, we have that $M$ is normal in $G$. Since $G$ is a Frobenius group, it follows that either $M \leq N$ or $N < M$. If $M \leq N$, then $L < N$, a contradiction. If $N < M$, then $G = N \rtimes L \leq M$, again a contradiction. Thus $L$ is a maximal subgroup of $G$.

It follows that $N$ is a minimal normal subgroup of $G$. Since $N$ is nilpotent by [2, Theorem 10.5.6], we can assume that $N = Z_p^m$, where $m \geq 1$ is a positive integer. Let $d$ be an element of $L$ of order $2$. Since $G$ is a Frobenius group and $N$ is abelian, we have $e^d = e^{-1}$ for every $1 \neq e \in N$. Thus $\langle e \rangle \rtimes \langle d \rangle$ is a subgroup of $G$.

We claim that $\langle e \rangle \rtimes \langle d \rangle$ is normal in $G$. Note that $|\langle e \rangle \times \langle d \rangle| = 2p$, where $p$ is an odd prime. If $\langle e \rangle \rtimes \langle d \rangle$ is not normal in $G$. Arguing as the subgroup $Q$, there exists a nonnormal subgroup $T$ of $G$ of nontrivial even order of largest order such that $\langle e \rangle \rtimes \langle d \rangle \leq T$. And we have that $T$ is also a Frobenius complement of $G$. But $N \cap T \geq \langle e \rangle \neq 1$, a contradiction. Thus $\langle e \rangle \rtimes \langle d \rangle$ is normal in $G$.

Since $\langle e \rangle \rtimes \langle d \rangle \not\leq N = Z_p^m$, we have $Z_p^m < \langle e \rangle \rtimes \langle d \rangle$. It follows that $m = 1$. Thus $N = Z_p$ is cyclic. By the N/C-theorem, since $C_G(N) = N$ it follows that $L$ is isomorphic to a subgroup of $\text{Aut}(Z_p) \cong Z_{p-1}$. Then $L$ is a cyclic group of order $2n$ for some $n > 1$. Assume that $L = \langle g \rangle$. We have $G = Z_p \rtimes \langle g \rangle$, where $p$ is an odd prime and $o(g) = 2n$ for some $n > 1$.

(2) Now we prove the sufficiency part.
(i) If all subgroups of $G$ of nontrivial even order are normal in $G$, then $G$ is obviously a EOTI-group.

(ii) Assume that $G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel $Z_p$ and complement $\langle g \rangle$, where $p$ is an odd prime and $o(g) = 2n$ for some $n > 1$. Let $S$ be any subgroup of $G$ of even order. It is obvious that $S \cap Z_p = 1$ or $Z_p$.

If $S \cap Z_p = 1$, we have $S \leq \langle g \rangle^f$ for some $f \in G$. Obviously $S$ is not normal in $G$. Then $N_G(S) = \langle g \rangle^f$. We have $S \cap S^x \leq \langle g \rangle^f \cap (\langle g \rangle^f)^x = 1$ for every $x \in G \setminus \langle g \rangle^f = G \setminus N_G(S)$.

If $S \cap Z_p = Z_p$, then $Z_p < S$. We have $S = S \cap G = S \cap (Z_p \rtimes \langle g \rangle) = Z_p \rtimes (S \cap \langle g \rangle)$. Then $S$ is normal in $G$.

It follows that $S$ is always a TI-subgroup of $G$ whenever $S \cap Z_p = 1$ or $Z_p$. Then $G$ is also a EOTI-group.

\[ \square \]

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