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Bifurcation Analysis and Dimension Reduction of a Predator-Prey Model for the L-H Transition

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The L-H transition denotes a shift to an improved confinement state of a toroidal plasma in a fusion reactor. A model of the L-H transition is required to simulate the time dependence of tokamak discharges that include the L-H transition. A 3-ODE predator-prey type model of the L-H transition is investigated with bifurcation theory of dynamical systems. The model is recognized as a slow-fast system.

INTRODUCTION

The L- and H-modes[1] are confinement states of a toroidal plasma, referring to states of low and high confinement, respectively. The transition from the L- to the H-mode is called the L-H transition. The T-mode is a transient, intermediate mode between the L- and H-mode, characterized by an oscillatory behavior. The L-H transition still lacks a first principle explanation. Modeling the L-H transition might contribute to a better understanding of the underlying mechanisms.

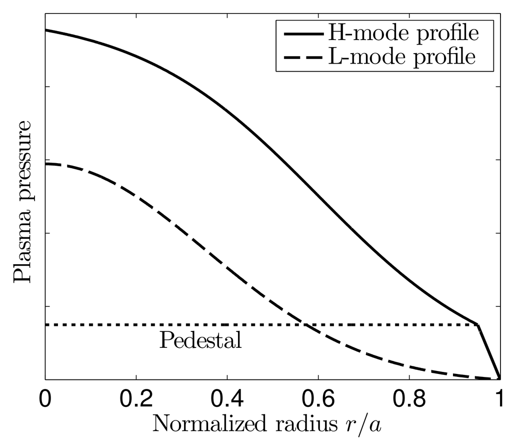

Figure: Sketch of the radial pressure profiles in L- and H-mode.

THE 3-ODE L-H TRANSITION MODEL

We consider the minimal 3-ODE predator-prey type L-H transition model suggested by Kim and Diamond [2, 3]. The model ignores spatial dependencies. The dependent variables in the model are:

- the drift wave turbulence level $\mathcal{E}$,
- the shear of the zonal flow $V_{\rm zf}$, and
- the gradient of the ion pressure $\mathcal{N}$.

The model can be formulated as

$$\frac{d}{dt}\mathcal{E} = \mathcal{E}\left(\mathcal{N} - a_1\mathcal{E} - a_2c_3^2\mathcal{N}^4 - a_3V_{\rm zf}^2\right),$$
$$\frac{d}{dt}V_{\rm zf} = V_{\rm zf}\left(\frac{b_1\mathcal{E}}{1+b_2c_3^2\mathcal{N}^4} - b_3\right),$$
$$\frac{d}{dt}\mathcal{N} = Q(t) - \mathcal{N}\left(c_1\mathcal{E} + c_2\right),$$

where a_i, b_i, c_i, $i = 1,2,3$ are parameters and Q is the heating power.

Introducing new variables and time,

$$u = a_1a_2^{1/3}c_3^{2/3}\mathcal{E}, \qquad v = a_2^{1/3}a_3c_3^{2/3}V_{\rm zf}^2,$$
$$w = a_2^{1/3}c_3^{2/3}\mathcal{N}, \qquad s = a_2^{-1/3}c_3^{-2/3}t,$$

results in the non-dimensionalized system

$$\dot{u} = u\left(w - u - v - w^4\right)$$
$$\dot{v} = \mu_1 v\left(\frac{u}{1+\mu_4w^4} - \mu_2\right)$$
$$\dot{w} = \mu_5\left(\sigma - w(1+\mu_3u)\right)$$

Here, μ_i, $i = 1,\dots,5$ are new parameters and σ is the rescaled heating power.

BIFURCATION ANALYSIS

The nullclines are

$$\mathcal{N}_u = \{u = 0\} \cup \{u = w\left(1-w^3\right) - v\},$$
$$\mathcal{N}_v = \{v = 0\} \cup \{u = \mu_2\left(1+\mu_4w^4\right)\},$$
$$\mathcal{N}_w = \{w(1+\mu_3u) = \sigma\}.$$

Stability of the equilibrium points:

- L is a stable node when it is below $\mathcal{N}_v$.
- H is always a saddle (unstable).
- T is a focus point. Stability depends on the value of σ.
- QH is stable for $\sigma > 1$.

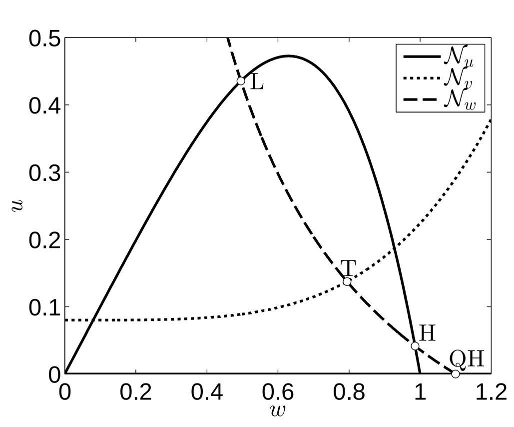

Figure: Nullclines and equilibrium points projected onto the uw-plane.

THREE TRANSITION TYPES

The bifurcation diagram structure depends on μ_i, $i = 1,\dots,5$. By varying μ_2 and μ_3 the three different transition types are observed.

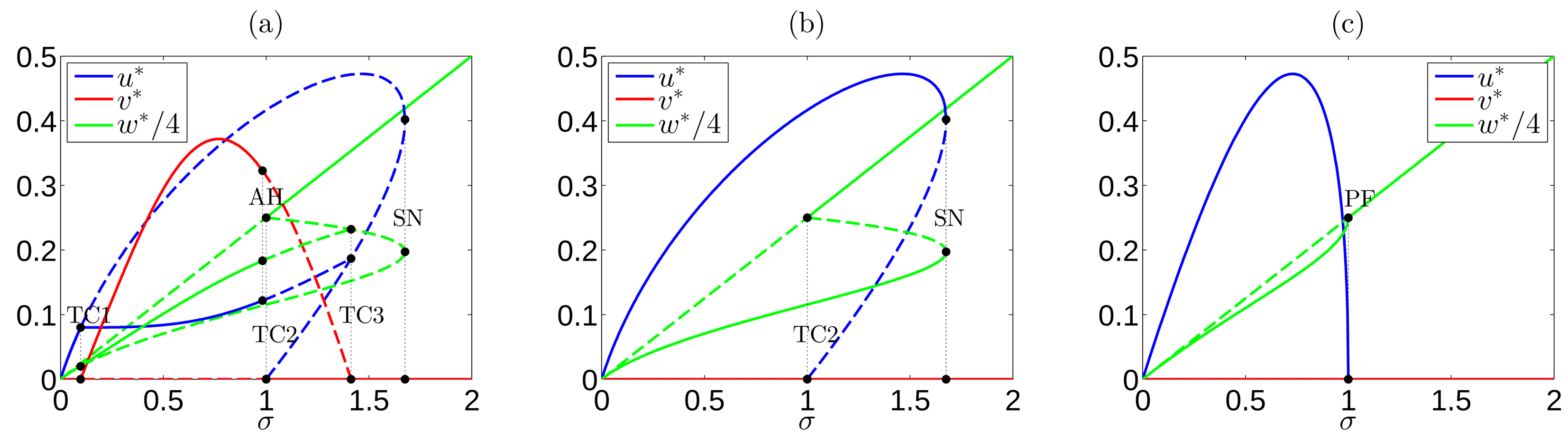

Figure: Bifurcation diagrams showing the three types of transitions: (a) an oscillating transtion, (b) a sharp transition allowing hysteresis, and (c) a smooth transition. Stable equilibria are shown as solid curves and unstable equilibria as dashed curves.

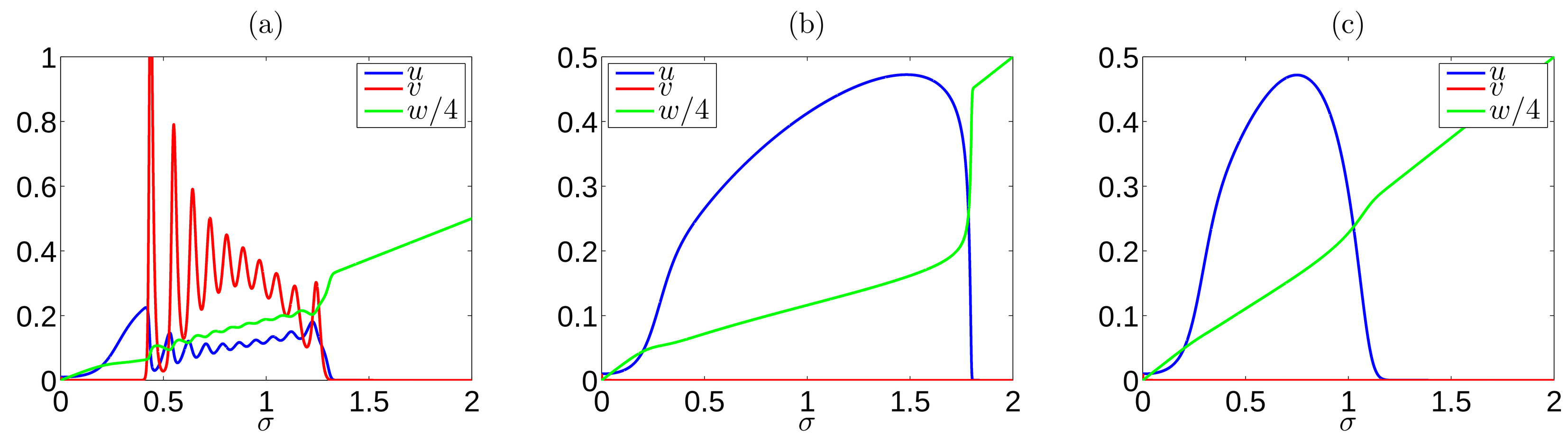

Figure: Numerical solutions showing the three types of transitions: (a) an oscillating, (b) a sharp, and (c) a smooth transition.

DIMENSION REDUCTION WITH GSPT

Put $\mu_5 = \frac{1}{\varepsilon}$, where $0 < \varepsilon \ll 1$.

- u and v are slow variables,
- w is a fast variable.

For $\varepsilon > 0$, but sufficiently small, solutions converge to the slow manifold,

$$\mathcal{M}_\varepsilon = \mathcal{M}_0 + \varepsilon\mathcal{M}_1 + \varepsilon^2\mathcal{M}_2 + \cdots$$

The reduced system of the flow is found by taking the limit $\varepsilon \to 0$:

$$\dot{u} = u\left(w - u - v - w^4\right)$$
$$\dot{v} = \mu_1 v\left(\frac{u}{1+\mu_4w^4} - \mu_2\right)$$
$$w = \frac{\sigma}{1+\mu_3u}$$

The reduced system contains the same dynamics as the full system.

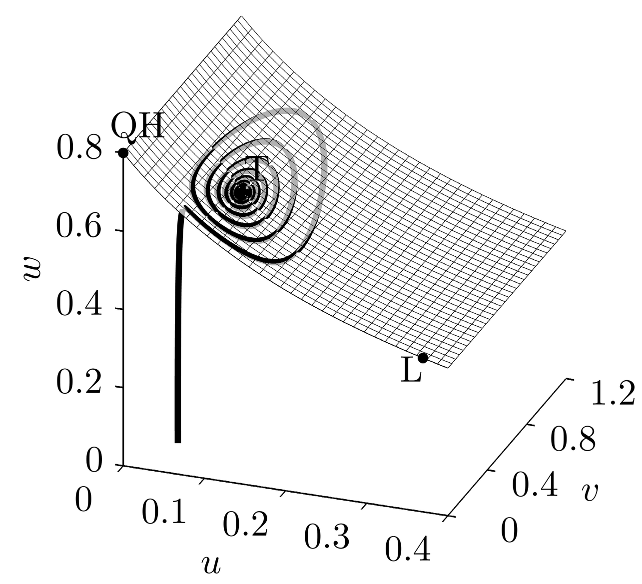

Figure: The critical manifold, $\mathcal{M}_0 = \{w = \frac{\sigma}{1+\mu_3u}\}$ and a numerical solution.

CONCLUSION

Kim and Diamond's 3-ODE L-H transition model was investigated with bifurcation theory.

- The model contains three types of transitions: an oscillating, a sharp with hysteresis, and a smooth transition.
- The system can be reduced to a 2-ODE system

A spatio-temporal L-H transition model has been proposed by Miki et al. [4].

References:

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